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ABSTRACT

This first unit in the SMSG's junior high mathematics series is a student text covering the following topics: what is mathematics?; numeration; whole numbers; non-metric geometry; factoring and primes; the rational number system; measurement; and area, volume, weight, and time. (DT)

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STUDENT'S TEXT UNIT NO. 1

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MATHEMATICS FOR JUNIOR HIGH SCHOOL

VOLUME 1

PART I

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SCHOOL MATHEMATICS STUDY GROUP

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YALE UNIVERSITY PRESS

School Mathematics Study Group

Mathematics for Junior High School, Volume 1

Unit 1

Mathematics for Junior High School, Volume I

Student's Text, Part I

Prepared under the supervision of the
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The preliminary edition of this volume was prepared at a writing session held at the University of Michigan during the summer of 1959, based, in part, on material prepared at the first SMSG writing session, held at Yale University in the summer of 1958. This revision was prepared at Stanford University in the summer of 1960, taking into account the classroom experience with the preliminary edition during the academic year 1959-60.

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FOREWORD

The increasing contribution of mathematics to the culture of the modern world, as well as its importance as a vital part of scientific and humanistic education, has made it essential that the mathematics in our schools be both well selected and well taught.

With this in mind, the various mathematical organizations in the United States cooperated in the formation of the School Mathematics Study Group (SMSG). SMSG includes college and university mathematicians, teachers of mathematics at all levels, experts in education, and representatives of science and technology. The general objective of SMSG is the improvement of the teaching of mathematics in the schools of this country. The National Science Foundation has provided substantial funds for the support of this endeavor.

One of the prerequisites for the improvement of the teaching of mathematics in our schools is an improved curriculum--one which takes account of the increasing use of mathematics in science and technology and in other areas of knowledge and at the same time one which reflects recent advances in mathematics itself. One of the first projects undertaken by SMSG was to enlist a group of outstanding mathematicians and mathematics teachers to prepare a series of textbooks which would illustrate such an improved curriculum.

The professional mathematicians in SMSG believe that the mathematics presented in this text is valuable for all well-educated citizens in our society to know and that it is important for the precollege student to learn in preparation for advanced work in the field. At the same time, teachers in SMSG believe that it is presented in such a form that it can be readily grasped by students.

In most instances the material will have a familiar note, but the presentation and the point of view will be different. Some material will be entirely new to the traditional curriculum. This is as it should be, for mathematics is a living and an ever-growing subject, and not a dead and frozen product of antiquity. This healthy fusion of the old and the new should lead students to a better understanding of the basic concepts and structure of mathematics and provide a firmer foundation for understanding and use of mathematics in a scientific society.

It is not intended that this book be regarded as the only definitive way of presenting good mathematics to students at this level. Instead, it should be thought of as a sample of the kind of improved curriculum that we need and as a source of suggestions for the authors of commercial textbooks. It is sincerely hoped that these texts will lead the way toward inspiring a more meaningful teaching of Mathematics, the Queen and Servant of the Sciences.

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PREFACE

Key ideas of junior high school mathematics emphasized in this text are: structure of arithmetic from an algebraic viewpoint; the real number system as a progressing development; metric and non-metric relations in geometry. Throughout the materials these ideas are associated with their applications. Important at this level are experience with and appreciation of abstract concepts, the role of definition, development of precise vocabulary and thought, experimentation, and proof. Substantial progress can be made on these concepts in the junior high school.

Fourteen experimental units for use in the seventh and eighth grades were written in the summer of 1958 and tried out by approximately 100 teachers in 12 centers in various parts of the country in the school year 1958-59. On the basis of teacher evaluations these units were revised during the summer of 1959 and, with a number of new units, were made a part of sample textbooks for grade 7 and a book of experimental units for grade 8. In the school year 1959-60, these seventh and eighth grade books were used by about 175 teachers in many parts of the country, and then further revised in the summer of 1960.

Mathematics is fascinating to many persons because of its opportunities for creation and discovery as well as for its utility. It is continuously and rapidly growing under the prodding of both intellectual curiosity and practical applications. Even junior high school students may formulate mathematical questions and conjectures which they can test and perhaps settle; they can develop systematic attacks on mathematical problems whether or not the problems have routine or immediately determinable solutions. Recognition of these important factors has played a considerable part in selection of content and method in this text.

We firmly believe mathematics can and should be studied with success and enjoyment. It is our hope that this text may greatly assist all teachers who use it to achieve this highly desirable goal.

Chapter 1

WHAT IS MATHEMATICS?

1-1. Mathematics as a Method of Reasoning

> "Once, on a plane, I fell into conversation with the man next to me. He asked me what kind of work I do. I told him I was a mathematician. He exclaimed, 'You are! Don't you get tired of adding columns of figures all day long?' I had to explain to him that this can best be done by a machine. My job is mainly logical reasoning."

Just what is this mathematics which many people are talking about these days? Is it counting and computing? Is it drawing figures and measuring them? Is it a language which uses symbols like a mysterious code? No, mathematics is not any one of these. It includes them but it is much more than all of these. Mathematics is a way of thinking, a way of reasoning. Some of mathematics involves experimentation and observation, but most of mathematics is concerned with deductive reasoning.

By deductive reasoning we prove that from certain given conditions, a definite conclusion necessarily follows. In arithmetic you have learned how to prove definite statements about numbers. If a classroom has 7 rows of seats with 5 seats in each row, then there are 35 seats in all. You know that this is true without counting the seats and without actually seeing the room.

Mathematicians use reasoning of this sort. They prove "if--then" statements. By reasoning they prove that if something is true, then something else must be true.

By logical reasoning you can often find all the different ways in which a problem can be solved. Sometimes you can show by reasoning that the problem has no solution. All the problems in Exercises 1-1 given below can be solved by reasoning. No calculations are needed, although you may find it helpful sometimes to draw a diagram. Can you solve them?

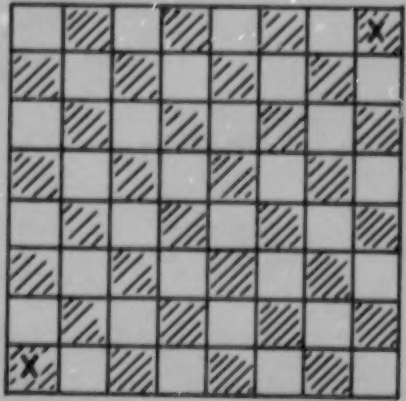
Exercises 1-1

1. A 200-pound man and his two sons each weighing 100 pounds want to cross a river. If they have only one boat and it can safely carry only 200 pounds, how can they cross the river?
2. If the man in Problem 1 weighs 175 pounds and one of his sons weighs 125 pounds and the other 100 pounds, can they use the same boat? If not, what weight must the boat carry safely so that they can cross the river?
3. A farmer wants to take a goose, a fox, and a bag of corn across a river. If the farmer is not with them, the fox will eat the goose or the goose will eat the corn. If the farmer has only one boat large enough to carry him and one of the others, how does he cross the river?
4. Is it possible to measure out exactly 2 gallons using only an 8 gallon container and a 5 gallon container? The containers do not have individual gallon markings or any other markings.
5. BRAINBUSTER. Three cannibals and three missionaries want to cross a river. They must share a boat which is large enough to carry only two people. At no time may the cannibals outnumber the missionaries, but the missionaries may outnumber the cannibals. How can they all get across the river using only the boat?
6. BRAINBUSTER. Eight marbles all have the same size, color, and shape. **Seven** of them have the same weight and the other is heavier. Using a balance scale, how would you find the heavy marble if you make only two weighings?

7. **BRAINBUSTER.** Suppose you have a checkerboard and dominoes. Each domino is just large enough to cover two squares on the checkerboard. How would you place the dominoes flat on the board in such a way as to cover all the board except two opposite corner squares?

Note: All the squares except the two squares in opposite corners are to be covered.

You may choose to leave the two white squares in opposite corners uncovered instead of the dark or shaded squares marked "x."



1-2. Deductive Reasoning

You can solve other types of problems by deductive reasoning. Suppose there are thirty pupils in your classroom. Can you prove that there are at least two of them who have birthdays during the same month? You can prove this in many ways without knowing the birthdays of any of them. One way is to reason like this. Imagine twelve boxes, one for each month of the year. Imagine also that your teacher writes each pupil's name on a slip of paper and then puts the slip into the proper box. If no box had more than one slip of paper, then there could not be more than 12 names in all. Since there are 30 names, then at least one box must contain more than one name.

Mathematicians are always eager to prove the best possible result. Here, you can use the same method to prove that at least three members of your class have birthdays in the same month. If no box had more than two slips of paper, then there could not be more than 24 names in all. Since there are 30 names, then at least three pupils have birthdays in the same month. Each problem in the next set of exercises can be solved by reasoning of the above type.

Exercises 1-2

1. Assume you have a set of five pencils which you are going to distribute among four of your classmates. Describe how one of them will get at least two pencils.
2. (a) Would you have to give at least two pencils to one person if you were distributing ten pencils among six persons?
   (b) What can you say if you are distributing a dozen pencils among five persons?
3. What is the least number of students that could be enrolled in a school so that you can be sure that there are at least two students with the same birthday?
4. What is the largest number of students that could be enrolled in a school so that you can be sure that they all have different birthdays?
5. There are five movie houses in a town. What is the smallest number of people that would have to go to the movies to make certain that at least two persons will see the same show?
6. In problem 5, what is the largest number of people that would have to go to the movies so that you could be sure that no two of them will see the same show?
7. If 8 candy bars are to be distributed among 5 boys, how many boys can receive three candy bars if each boy is to receive at least one bar?

8. In a class of 32 students various committees are to be formed. No student can be on more than one committee. Each committee contains from 5 to 8 students. What is the largest number of committees that can be formed?
9. BRAINBUSTER. What is the answer to problem 8 if every student can be on either one or two committees?

1-3. From Arithmetic to Mathematics

Another way in which mathematicians and other scientists solve problems is to make experiments and observations. This method is called the experimental method. Can you think of scientific problems which have been solved in this way?

Many times in mathematics we experiment to discover a general way of solving an entire set of problems. After the general method has been discovered, we try to prove that it is correct by logical reasoning.

The part of mathematics which you know best is arithmetic. Often in arithmetic you can obtain results by experiment and by reasoning which can save you a lot of hard work and time spent in calculation.

When Karl Friedrich Gauss, a famous mathematician, was about 10 years old, his teacher wanted to keep the class quiet for a while. He told the children to add all the numbers from 1 to 100, that is $1 + 2 + 3 + \dots + 100$. (Note: To save writing all the numbers between 3 and 100, it is customary to write three dots. This may be read "and so on.") In about two minutes Gauss was up to mischief again. The teacher asked him why he wasn't working on the problem. He replied, "I've done it already." "Impossible!" exclaimed the teacher. "It's easy," answered Gauss. "First I wrote:

1 + 2 + 3 + 4 + ... + 100, then I wrote the numbers in reverse order:

$\underline{100} + \underline{99} + \underline{98} + \underline{97} + \ldots + \underline{\ \ 1}$, then I added each pair of numbers:

101 + 101 + 101 + 101 + ... + 101.

When I added, I got one hundred 101's. This gave me 100 x 101 = 10,100. But I used each number twice. For example, I added 1 to 100 at the beginning, and then I added 100 to 1 at the end. So I divided the answer 10,100 by 2. The answer is $\frac{10,100}{2}$ or 5,050."

Who was Karl Friedrich Gauss? When did he live? You may find answers to these questions in your encyclopedia.

Exercises 1-3

1. Add all the numbers from 1 to 5, that is, 1 + 2 + 3 + 4 + 5, using Gauss's method. Can you discover another short method different from the Gauss method?
2. Can Gauss's method be applied to the problem of adding the numbers: 0 + 2 + 4 + 6 + 8?
3. By a short method add the odd numbers from 1 to 15, that is 1 + 3 + 5 + ... + 15. (To save writing the odd numbers between 5 and 15 it is customary to write three dots. This may be read "and so on.")
4. This problem gives you a chance to make another discovery in mathematics.

	Add the numbers below:		Multiply the numbers below:	
a.	1 + 3	= ?	2 × 2	= ?
b.	1 + 3 + 5	= ?	3 × 3	= ?
c.	1 + 3 + 5 + 7	= ?	4 × 4	= ?

   d. Look at the sums and the products on the right. What seems to be the general rule for finding the sums of numbers on the left?

   e. Apply your new rule to 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15. Check with your answer in Exercise 3.

5. Add the odd numbers: 7 + 9 + 11 + ... + 17.

6. Add the even numbers from 22 to 40: 22 + 24 + 26 + ... + 40.
7. Add all the numbers from 0 to 50: 0 + 1 + 2 + ... + 50.
8. Will Gauss's method work for any series of numbers whether we start with 0 or 1, or any other number?
9. BRAINBUSTER. Add all the numbers from 1 to 200 by Gauss's method. Then add all the numbers from 0 to 200 by Gauss's method. Are the answers for these two problems the same? Why?
10. BRAINBUSTER. Suppose Gauss's teacher had asked him to add the numbers: 1 + 2 + 4 + 8 + 16 + ... + 256. (Here we start with 1 and double each number to get the one which follows.) Is there any short method for getting the sum?
11. BRAINBUSTER. If you have a short method for doing Problem 10, try this one. Add the numbers: 2 + 6 + 18 + ... + 486, where we start with 2 and multiply each number by 3 to get the one which follows. The answer is 728. Can you find a short method for getting the sum?

1-4. Kinds of Mathematics

Mathematicians reason about all sorts of puzzling questions and problems. When they solve a problem they usually create a little more mathematics to add to the ever-increasing stockpile of mathematical knowledge. The new mathematics can be used with the old to solve even more difficult problems. This process has been going on for centuries and the total accumulation of mathematics is far greater than most people can imagine. Arithmetic is one kind of mathematics. The trigonometry, algebra, and plane geometry you will study are other kinds.

Today there are more than eighty different kinds of mathematics. No single mathematician can hope to master more than a small bit of it. Indeed the study of any one of these eighty different branches would occupy a mathematical genius throughout his entire life. So don't be surprised if your teacher sometimes fails to know all the answers!

Moreover, hundreds of pages of new mathematics are being created every day of the year -- much more than one person could possibly read in the same day. In fact, in the past 50 years, more mathematics has been discovered than in all the preceding thousands of years of man's existence.

Chance or Probability

One of the eighty kinds of mathematics which is interesting to mathematicians and also useful in present day problems is the study of chance or probability.

Many happenings in our lives are subject to uncertainty, or chance. There is a chance the fire alarm may ring as you read this sentence. There is a chance that the teacher may give a test today. It is difficult to predict exactly the chance of any of these events, although in such cases we would be satisfied to know that the chance is very small.

Many times, however, we would like to calculate the chance more precisely in order to compare the chances of several alternatives. Mathematicians have been studying such questions for many years, and this kind of mathematics is called probability.

Disney Needed the Mathematicians' Help

A number of such probability questions were answered for Walt Disney before Disneyland was built. When he considered setting up Disneyland, Disney wanted to know how big to build it, where to locate it, what admission to charge, and what special facilities to provide for holidays. He didn't want to take a chance on spending $17,000,000 to build Disneyland without knowing something of the probability of success.

What he really wanted answered was this type of question: If I build a certain type of facility, at this particular location, and charge so much admission, then what is the probability that I will make a certain amount of money?

Disney went to the Stanford Research Institute. There he talked with a group of mathematically trained people who are specialists in applying mathematical reasoning to business problems.

The people at Stanford first collected statistics about people--(their income, travel habits, amusement preferences, number of children, etc.). Combining this information by mathematical reasoning they predicted the probability that people would come to a certain location and pay a given price of admission. From reasoning like this they could predict the probability of having a successful Disneyland of a certain type at a given spot. Knowing the chance of success under given conditions, Disney was better able to decide how and where to build Disneyland and how much to charge for admission.

This example is typical of the way probability is often used to give an estimate of the degree of uncertainty of an event or the chance of success of a proposed course of action.

The following problems are mainly to give you some idea of what simple probability is about.

Exercises 1-4

1. To see how a mathematician might begin to think about probability, imagine that you toss two pennies. There are four equally likely ways that the coins can come up:

First Penny	H	H	T	T
Second Penny	H	T	H	T

We are using H to represent heads and T to represent tails. HH describes the event of both coins coming up heads. We say then that the probability of tossing 2 heads with 2 coins is 1 out of 4 or $\frac{1}{4}$. We cannot predict what will happen in any one toss, but we can predict that if the two coins are tossed 100,000 times or so, then the two heads will come up about $\frac{1}{4}$ of the time. Try this experiment 100 times with some of your classmates and tabulate your results. Tabulate other experiments of the whole class and see how many HH trials turn up in the total number of trials.

2. What is the probability that both coins show tails when two coins are tossed?
3. What is the probability that a head appears when one coin is tossed?
4. What is the probability of drawing the ace of spades from a full deck of 52 cards?
5. What is the probability of drawing some ace from a full deck of 52 cards?
6. What is the probability of throwing a die (one of a pair of dice) and having two dots come up?
7. There are four aces (from a playing card deck of 52 cards) to be dealt to four people. What is the probability that the first person who receives a card gets the ace of hearts?
8. BRAINBUSTER. What is the probability of throwing a pair of ones (one dot on each die) with one pair of dice?
9. BRAINBUSTER. What is the probability of having three heads come up if 3 pennies are tossed? What is the probability of having exactly 2 heads turn up? At least 2 heads?

1-5. Mathematics Today

You are living in a world which is changing very rapidly. To get some idea of the changes in the past 20 years, ask your parents what life was like in their junior high school days. Did you realize that such things as color TV, atomic submarines, jet planes, and space satellites are all recent developments? There are new medicines and vaccines. There are new ways to make business decisions. There are new ways of computing. And, there are hundreds of other new developments reported every day. Can you think of some other new discoveries, new developments and new products? The interesting thing is that mathematics and mathematicians have had a part in almost all of them.

In the telephone industry mathematics is used to design switching devices, so that when you dial a phone number you have a good chance of avoiding a busy signal. Mathematics has contributed especially in discovering better ways to send information over telephone wires or by wireless communication.

In the aircraft industry mathematics helps determine the best shape for an airplane or space ship, and how strong its construction must be. Another kind of mathematics predicts whether a plane will shake itself to pieces as it flies through stormy air at high speeds. Still different forms of mathematics help design the radio and radar devices used to guide the plane and to communicate with other planes and with airfields.

In almost all kinds of manufacturing, mathematics (the probability you studied in the last section) is used to predict the reliability of the goods manufactured. Many times the manufacturer must make a guarantee of reliable performance based on a mathematical prediction. If the mathematician makes a mistake the manufacturer loses money (and the mathematician may lose his job!).

Other kinds of new mathematics help business men decide how much to produce, how best to schedule production to avoid paying too much overtime, and where to build new plants in order to reduce shipping costs.

In the petroleum industry mathematics is used extensively in deciding how many oil wells to sink and where to drill to get the most oil from an oil field at the least cost. Mathematical techniques also help the gasoline manufacturer to decide how much gasoline of various qualities to refine from different crude petroleums.

In all these businesses and industries, in universities and government agencies as well, the mathematics of computing and the big new electronic computers are widely used.

Why is mathematics now used in so many areas? One reason is that mathematical reasoning, and the kinds of mathematics already developed, provide a precise way to describe complicated situations and to analyze difficult problems. The language of mathematics is expressed in shorthand symbols, all precisely defined, and used according to definite logical rules. This often makes it possible to study problems too complicated to visualize. Frequently, mathematical reasoning predicts the possibility or impossibility of a scientific experiment. Sometimes, the most useful answer a mathematician can find is to prove beyond question that the problem (or machine, or system, or experiment) being studied is impossible. The mathematical work may also show why the problem is impossible in the present form and may suggest a way to get around the difficulties.

1-6. Mathematics as a Vocation

Before World War II almost all mathematicians were employed as teachers in schools and colleges. Since then, the world of mathematics and the world of mathematicians have changed tremendously. Today there are more teachers of mathematics than ever before. In junior and senior high school there are about 50,000 people who teach mathematics. There are about 5,000 more teachers employed in colleges and universities. But now (1960), in business, industry, and government there are more than 20,000 persons working as mathematicians.

The Federal Government hires mathematicians in numerous agencies for many different assignments. Literally thousands of people work with computers and computer mathematics. Industries of all types are hiring mathematicians to solve complex mathematical problems, to help other workers with mathematical difficulties and even to teach mathematics to other employees.

These changes have been brought about by the revolutionary advances in science and technology which we talked about. These changes are still continuing. By the time you are ready for a job, opportunities for a career in mathematics will be even more numerous and varied.

You will find a great deal about mathematics and its role in present-day life in the current news. The activities in section

1-6 and section 1-7 suggest some ways of finding this information.

You should start your search now and continue collecting materials throughout the year. Many items will be suitable for your class bulletin board.

Class Activities 1-6

1. Make a list of businesses and industries that hire mathematicians and computers. You can do this by collecting want ads for mathematicians from newspapers, mathematical and technical magazines.
2. Collect Civil Service folders describing opportunities for mathematicians in government service.
3. Look in your newspapers, in weekly magazines, and in scientific magazines for feature stories about mathematicians and mathematics.

1-7. Mathematics in Other Vocations

Many people who are not primarily mathematicians need to know a lot of mathematics, and use it almost every day. This has long been true of engineers and physicists. Now they find it necessary to use even more advanced mathematics. Every new project in aircraft, in space travel, or in electronics demands greater skills from the engineers, scientists, and technicians.

Mathematics is now being widely used and required in fields such as social studies, medical science, psychology, geology, and business administration. Mathematical reasoning and many kinds of mathematics are useful in all these fields. Much of the use of electronic computers in business and industry is carried on by people who must learn more about mathematics and computing in order to carry on their regular jobs. To work in many such jobs you are required to know a lot about mathematics. Merely to understand these phases of modern life, and to appreciate them enough to be a good citizen, you will need to know about mathematics.

This is one of the major reasons this textbook is being written -- we know that you will need far more mathematics in your lifetime than your parents or your teachers ever were required to learn. We hope to be able to give you a foundation for all your future studies in mathematics and other sciences and at the same time share some of the fun and excitement that people enjoy in discovering and using mathematics.

Class Activities 1-7

1. Extend your search of newspapers and magazines to include job opportunities and feature stories about people using mathematical training in other vocations.
2. Look in Civil Service folders to find the mathematical requirements for Civil Service appointments as engineer, physicist and statistician.
3. Find out the requirements for some of the following vocations: (Your guidance counselor may be able to help you with guidance pamphlets or college catalogues.)

    Accountant
    Actuary
    Aeronautical Engineer
    Agricultural Engineer
    Bricklayer
    Carpenter
    Chemical Engineer
    Chemist
    Doctor
    Economist
    Electrical Engineer
    Electrician
    Farmer
    Machinist
    Mathematician
    Nurse
    Physicist
    Plumber
    Programmer
    Psychologist
    Statistician

1-8. Mathematics for Recreation

Just as music is the art of creating beauty with sounds, and painting is the art of creating beauty with colors and shapes, so mathematics is the art of creating beauty with combinations of ideas. Many people enjoy mathematics as a fascinating hobby. Many people study mathematics for fun just as other people enjoy music or painting for pleasure.

For many thousands of years people have enjoyed working various kinds of problems. A good example of this is a problem concerning the Königsberg Bridges. In the early 1700's the city of Königsberg, Germany was connected by seven bridges. Many people in the city at that time were interested in finding if it were possible to walk through the city so as to cross each bridge exactly once.

From the diagram showing these bridges, can you figure out whether or not this can be done? You may be interested in knowing that a mathematician answered this question in the year 1736.

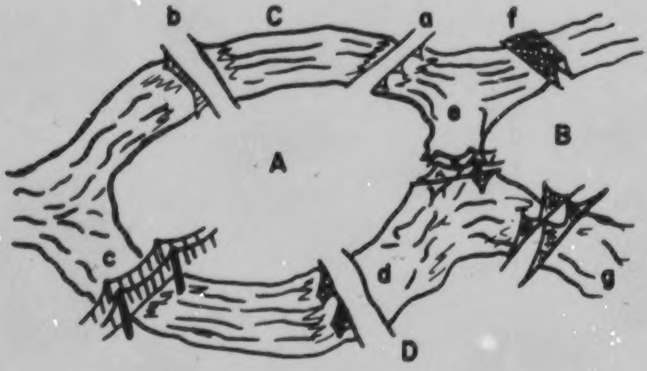


Figure 1-8a

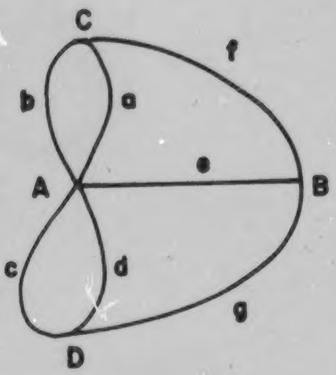


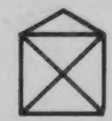
Figure 1-8b

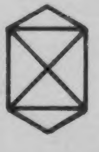
Figure 1-8b will help you see the various ways of walking through the city using the bridges to go from one piece of land to another. Use C in place of the piece of land to the north and D, the land to the south. A is the island and B is the land to the east. The lines leading from A, C, D, and B show routes across the bridges to the various parts of the city. The bridges are lettered a, b, c, d, e, f, g. At points B, C, and D three routes come together and at point A five routes meet.

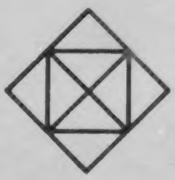
Several kinds of mathematics were developed by mathematicians as a result of their work on such thought-provoking problems as this one.

Exercise 1-8

1. BRAINBUSTER. Of the three figures shown, two can be drawn without lifting your pencil or retracing a line, while the other cannot. Which two can be drawn without going over any lines twice and without crossing any lines?



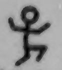
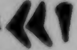




1-9. Highlights of First Year Junior High School Mathematics

During this year you will develop a better understanding of what mathematics really is. You will have many opportunities to experiment with mathematics and use deductive reasoning. Though mathematics is much more than just counting, computing, measuring, and drawing, you will use many operations and applications in the following chapters. The next paragraphs will give you an over-all picture of the various topics you will be exploring.

You will explore records of number from the primitive peoples' scratches in the dirt to written symbols for numbers.

The Egyptian symbol  (astonished man) for 1,000,000 and the Babylonian symbols  for 21 give you an idea of what is to come in the next chapter. You will find that the numeral 100 (read one, zero, zero) does not always represent one hundred. Does this surprise you?

For many years you have used counting numbers, such as 1, 2, 3, 4, and so on. Are there other kinds of numbers? Yes, you will become acquainted with several other kinds.

If you notice how numbers behave when you add or multiply them you will find some properties that are always true in addition and multiplication. Zero and one also have special properties which you may have discovered. This year you will be observing numerals much more closely than you have ever done. For some of you it will be similar to looking through a magnifying glass. When you really look at a problem carefully, you discover how much clearer the mathematics in the problem becomes. Ask yourself, "What has happened to these numbers? Can I tell what seems to be true?"

For many years you have used the word "equal" and know a symbol for it. Are things always equal? Could you suggest a symbol for "is not equal to?" You are already familiar with many other symbols used in mathematics. Some of them you have used so often that you use them without thinking much about them. Look at the symbol $\frac{23}{30}$. Are you familiar with this fraction? Now look at an Egyptian way of writing this fraction many years ago: $\frac{1}{3} + \frac{1}{6} + \frac{1}{5} + \frac{1}{15}$. Do you agree that the symbol $\frac{23}{30}$ is much simpler and easier to handle? New symbols will be introduced this year. They should help you to be more precise in your ordinary speech.

Another interesting part of your year will be spent considering ideas about points, lines, planes, and space. You may already have some ideas about these. Have you ever built models? If you have, you will have some of your own ideas about points, lines, planes, and space. These ideas will need to be carefully considered and expanded. You may not agree with your first observations after you have studied more about these ideas.

Can you distinguish between quantities that are counted and quantities that are measured? Different kinds of numbers are needed for counting and measuring, but you will already have studied about these numbers and be ready to use them in measurement of various types of things. Your shop and home economics teachers will be good judges of how well you can do this.

We cannot possibly tell you all about your first year in junior high school mathematics, or what mathematics is, in just one chapter. However, we hope that as you study mathematics this year you will gain a much better idea of what mathematics is and why you should know as much of it as you can learn.

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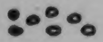
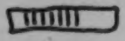
Chapter 2

NUMERATION

2-1. Caveman's Numerals

In primitive times, boys and girls of your age were probably aware of simple numbers in counting, as in counting "one deer" or "two arrows." Primitive peoples learned to keep records of numbers. Sometimes they tied knots in a rope, or used a pile of pebbles, or cut marks in sticks to count objects. A boy counting sheep would have  pebbles, or he might make notches in a stick, as  . One pebble, or one mark in the stick would represent a single sheep. He could tell several days later that a sheep was missing if there was not a sheep for each pebble, or for each mark on the stick. You make the same kind of record when you count votes in a class election, one mark for each vote, as 𝍸 ||. When people began to make marks for numbers, by making scratches on a stone or in the dirt, or by cutting notches in a stick, they were writing the first numerals. Numerals are symbols for numbers. Thus the numeral "7" is a symbol for the number seven. Numeration is the study of how symbols are written to represent numbers.

As centuries passed, early people used sounds, or names, for numbers. Today we have a standard set of names for numbers. A rancher counting sheep compares a single sheep with the name "one," and 2 sheep with the name "two," and so on. Man now has both symbols (1, 2, 3, . . .) and words (one, two, three, . . .) which may be used to represent numbers.

Ancient Number Systems

One of the earliest systems of writing numerals of which we have any record is the Egyptian. Their hieroglyphic, or picture, numerals have been traced as far back as 3300 B.C. Thus, about 5000 years ago, Egyptians had developed a system with which they could express numbers up to millions. Egyptian symbols are shown on the following page.

Our Number	Egyptian Symbol	Object Represented
1	𓏺	stroke or vertical staff
10	𓎆	heel bone
100	𓍢	coiled rope or scroll
1,000	𓆼	lotus flower
10,000	𓂭	pointing finger
100,000	𓆐	burbot fish (or polliwog)
1,000,000	𓁨	astonished man

These symbols were carved on wood or stone. The Egyptian system was an improvement over the caveman's system because it used these ideas:

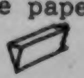
1. A single symbol could be used to represent the number of objects in a collection. For example, the heel bone represented the number ten.

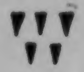
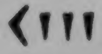
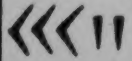
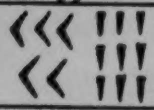
2. Symbols were repeated to show other numbers. The group of symbols 𓍢𓍢𓍢 meant 100 + 100 + 100 or 300.

3. This system was based on groups of ten. Ten strokes make a heelbone, ten heelbones make a scroll, and so on.

The following table shows how Egyptians wrote numerals:

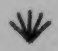
Our numerals	4	11	23	20,200	1959
Egyptian numerals	𓏺𓏺𓏺𓏺	𓎆𓏺	𓎆𓎆𓏺𓏺𓏺	𓂭𓂭𓍢𓍢	𓆼 𓍢𓍢𓍢 𓎆𓎆𓎆 𓏺𓏺𓏺 𓍢𓍢𓍢 𓎆𓎆 𓏺𓏺𓏺 𓍢𓍢𓍢 𓏺𓏺𓏺

About 4000 years ago, around 2000 BC, the Babylonians lived in the part of Asia we now call the Middle East. They did their writing with a piece of wood on clay tablets. These tablets are called cuneiform tablets. Clay was used because they did not know how to make paper. The pieces of wood were wedge-shaped at the ends as  . A drawing instrument of this type is called a stylus. With the stylus a mark ▼ was made on the clay to represent the number "one." By turning the stylus, they made this symbol < for "ten." They combined these symbols to write numerals up to 59 as shown in the table below:

Our numerals	5	13	32	59
Babylonian numerals				

Later in this chapter you will learn how the Babylonians wrote numerals, or symbols, for numbers greater than 59.

The Roman system was used for hundreds of years. There are still a few places at the present time where these numerals are used. Dates on cornerstones and chapter numbers in books are often written in Roman numerals.

Historians believe that the Roman numerals came from pictures of fingers, like this: |, ||, |||, and ||||. The Romans then used a hand for five,  . Gradually some of the marks were omitted, and they wrote V for five. Two fives put together made the symbol for ten, X. The other symbols were letters of their alphabet. The following table shows the other letters used by the Romans:

Our Numeral	1	5	10	50	100	500	1000
Roman Numeral	I	V	X	L	C	D	M

In early times the Romans repeated symbols to make larger numbers in the same way that the Egyptians had done many years before. Later, the Romans made use of subtraction to shorten some numbers.

The values of the Roman symbols are added when a symbol representing a larger quantity is placed to the left in the numeral.

MDCLXVI = 1,000 + 500 + 100 + 50 + 10 + 5 + 1 = 1666.

DLXI = 500 + 50 + 10 + 1 = 561

When a symbol representing a smaller value is written to the left of a symbol representing a larger value, the smaller value is subtracted from the larger.

$$IX = 10 - 1 = 9.$$
$$XC = 100 - 10 = 90.$$

The Romans had restrictions on subtracting.

1. V, L and D (symbols representing numbers that start with 5) are never subtracted.
2. A number may be subtracted only in the following cases:
   I can be subtracted from V and X only.
   X can be subtracted from L and C only.
   C can be subtracted from D and M only.

Addition and subtraction can both be used to write a number. First, any number whose symbol is placed to show subtraction is subtracted from the number to its right; second, the values found by subtraction are added to all other numbers in the numeral.

$$CIX = 100 + (10 - 1) = 100 + 9 = 109.$$
$$\begin{aligned} MCMLX &= 1{,}000 + (1{,}000 - 100) + 50 + 10 \\ &= 1{,}000 + 900 + 50 + 10 = 1960. \end{aligned}$$

Sometimes the Romans wrote a bar over a number. This multiplied the value of the symbol by 1,000.

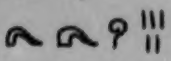
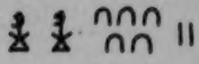
$\overline{X} = 10{,}000$, $\overline{C} = 100{,}000$, and $\overline{XXII} = 22{,}000$.

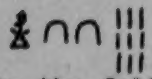
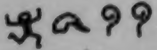
There were many other number systems used throughout history: the Korean, Chinese, Japanese, and Indian systems in Asia; the Mayan, Incan, and Aztec systems of the Americas; the Hebrew, Greek, and Arabian systems in the Mediterranean regions. You might enjoy looking up one of these. If you do, you will find the study of these various systems very interesting. We do not have the time needed to discuss all of them in this chapter.

Exercises 2-1

1. Write the following numbers using Egyptian numerals:
   a. 19 b. 53 c. 666 d. 1960 e. 1,003,214
2. The Egyptians usually followed a pattern in writing large numbers. However, the meanings of their symbols were not changed if the order in a numeral was changed. In what different ways can twenty-two be written in Egyptian notation?

3. Write our numerals for each of the following numbers:

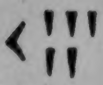
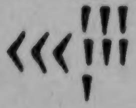
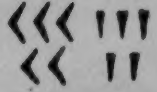
a.  b. 

c.  d. 

4. Write the following numbers in Babylonian numerals:

a. 9 b. 22 c. 51

5. Write our numerals for each of the following numbers:

a.  b.  c. 

6. Write our numerals for each of the following numbers:

a. XXIX b. LXI c. XC d. CV e. DCLXVI f. $\overline{D}$

g. MCDXCII

7. Write the following numbers in Roman numerals:

a. 19 b. 57 c. 888 d. 1690 e. 1,000,000

f. 15,000

8. a. How many different symbols were used in writing Egyptian numerals?

   b. How many different symbols were used in writing the Babylonian numerals for numbers up to 59?

   c. How many different symbols did the Romans use in writing numerals?

   d. How many different symbols are there in our system?

9. a. Do XC and CX have the same meaning when written in Roman notation? Explain, using our numeral or numerals.

   b. Was the position of a symbol in a numeral important in the later Roman system? If so, what does the position of the numeral show?

10. a. What number is represented by III in the Roman system?

   b. What number is represented by 111 in our system?

   c. Can you explain why your answers are different for parts a and b?

11. Write each of these pairs of numbers in our notation, then add the results and change the answer to Roman numerals.
    a. MDCCIX and DCLIV
    b. MMDCXL and MCDVIII

2-2. The Decimal System

History and Description

All of the early number systems are an improvement over matching notches or pebbles. It is fairly easy to represent a number in any of them. It is difficult to use them to add and multiply. Some instruments, like the abacus, were used by ancient peoples to do arithmetic problems.

The way we write numerals was developed in India. Arab scholars learned about these numerals and carried them to Europe. Because of this, our numerals are called Hindu-Arabic numerals. It is interesting to note that most Arabs have never used these symbols. Because our system uses groups of ten, it is called a decimal system. The word decimal comes from the Latin word "decem," which means "ten."

The decimal system is used in most of the world today because it is a better system than the other number systems discussed in the previous section. Therefore, it is important that you understand the system and know how to read and write numerals in this system.

Long ago man learned that it was easier to count large numbers of objects by grouping the objects. We use the same idea today when we use a dime to represent a group of ten pennies, and a dollar to represent a group of ten dimes. Because we have ten fingers it is natural for us to count by tens. We use ten symbols for our numerals. These symbols, which are called digits, are 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0. The word digit refers to our fingers and to these ten number symbols. With these ten symbols we can write a number as large or as small as we wish.

The decimal system uses the idea of place value to represent the size of a group. The size of the group represented by a symbol depends upon the position of the symbol or digit in a numeral. The symbol tells us how many of that group we have. In the numeral 123, the "1" represents one group of one hundred; the "2" represents two groups of ten, or twenty; and the "3" represents three ones, or three. This clever idea of place value makes the decimal system the most convenient system in the world.

Since we group by tens in the decimal system, we say its base is ten. Because of this, each successive (or next) place to the left represents a group ten times that of the preceding place. The first place tells us how many groups of one. The secondplace tells us how many groups of ten, or ten times one (10 × 1). The third place tells us how many groups of ten times ten (10 × 10), or one hundred; the next, ten times ten times ten (10 × 10 × 10), or one thousand, and so on. By using a base and the ideas of place value, it is possible to write any number in the decimal system using only the ten basic symbols. There is no limit to the size of numbers which can be represented by the decimal system.

To understand the meaning of the number represented by a numeral such as 123 we add the numbers represented by each symbol. Thus 123 means (1 × 100) + (2 × 10) + (3 × 1), or 100 + 20 + 3. The same number is represented by 100 + 20 + 3 and by 123. When we write a numeral such as 123 we are using number symbols, the idea of place value, and base ten.

One advantage of our decimal system is that it has a symbol for zero. Zero is used to fill places which would otherwise be empty and might lead to misunderstanding. In writing the numeral for three hundred seven, we write 307. Without a symbol for zero we might find it necessary to write 3-7. The meaning of 3-7 or 3 7 might be confused. The origin of the idea of zero is uncertain, but the Hindus were using a symbol for zero about 600 A.D., or possibly earlier.

The clever use of place value and the symbol for zero makes the decimal system one of the most efficient systems in the world. Pierre Simon Laplace (1749 - 1827), a famous French mathematician, called the decimal system one of the world's most useful inventions.

Reading and Writing Decimal Numerals

Starting with the first place on the right, each place in the decimal system has a name. The first is the units place, the second the tens place, the third the hundreds place, the fourth the thousands place, and so on. The places continue indefinitely. Do not confuse the names of our ten symbols with the names of the places. Long numbers are easier to read if the digits are separated at regular intervals. Starting at the right, every group of three digits is separated by a comma. These groups also have names as shown by the following table for the first four groups of digits.

Group Name	Billion			Million			Thousand			Unit		
Place Name	Hundred Billion	Ten Billion	Billion	Hundred Million	Ten Million	Million	Hundred Thousand	Ten Thousand	Thousand	Hundred	Ten	Unit
Digits	5	4	5,	4	6	5,	7	3	8,	9	2	1

The names of the digits, the concept of place value and the group name are all used to read a number. To read the numeral shown in the table we start with the group on the left, reading the number represented by the first group of digits as one numeral. This is followed by the name of the group, as "five hundred forty-five billion." Then we read each of the following groups, using the name for each group as shown in the table, except that we do not use the name "unit" in reading the last group on the right. The whole numeral shown in the table is correctly read as "five hundred forty-five billion, four hundred sixty-five million, seven hundred thirty-eight thousand, nine hundred twenty-one." The word "and" is not used in reading numerals for whole numbers.

Although we have only ten symbols, we use these symbols again and again. They are used in different positions in numerals to express different numbers. Similarly, in reading numerals we use only a few basic words. We use the names for the symbols, "one, two, three, four," and so on. Then we have the words "ten, eleven, twelve, hundred, thousand," and so on. For other names we use combinations of names, as in "thirteen" which is "three and ten," or in "one hundred twenty-five" which is "one hundred, two tens, and five ones.

Exercise 2-2

1. How many symbols are used in the decimal system of notation? Write the symbols.
2. Write the names for the first nine places in the decimal system. Begin with the smallest place and keep them in order, as "one, ten, ?, ?, . . .
3. Practice reading the following numerals orally, or write the numerals in words.
    a. 300
    b. 3005
    c. 7109
    d. 15,015
    e. 234,000
    f. 608,014
    g. 100,009
    h. 430,001
    i. 999,999
4. Practice reading the following numerals orally, or write the numerals in words.
    a. 7,036,298
    b. 9,300,708,500,000
    c. 20,300,400,500
    d. 900,000,000,000
5. Write the following using decimal numerals:
    a. one hundred fifty-nine.
    b. five hundred three.
    c. six thousand, eight hundred fifty-seven.

d. three million, seventy thousand, thirteen.
e. four billion, three hundred seventy-six million, seven thousand.
f. twenty thousand, ten.
g. nine million, fifteen thousand, two hundred.

6. a. Write the numeral representing the largest five place number in the decimal system.
   b. Explain what this number means just as $(3 \times 10) + (5 \times 1)$ explains the meaning of 35.
   c. Write the numeral in words.

7. a. Write the numeral representing the smallest 6-place number in the decimal system.
   b. Explain the meaning of this number.
   c. Write the numeral in words.

2-3. Expanded Numerals and Exponential Notation

We say that the decimal system of writing numerals has a base ten. Starting at the units place, each place to the left has a value ten times as large as the place to its right. The first six places from the right to the left are shown below:

hundred thousand	ten thousand	thousand	hundred	ten	one
$(10\times10\times10\times10\times10)$	$(10\times10\times10\times10)$	$(10\times10\times10)$	(10×10)	(10)	(1)

Often we write these values more briefly, by using a small numeral to the right and above the 10. This numeral shows how many 10's are multiplied together. Numbers that are multiplied together are called factors. In this way, the values of the places are written and read as follows:

$(10 \times 10 \times 10 \times 10 \times 10)$	10^5	"ten to the fifth power"
$(10 \times 10 \times 10 \times 10)$	10^4	"ten to the fourth power"
$(10 \times 10 \times 10)$	10^3	"ten to the third power"
(10×10)	10^2	"ten to the second power"

(10)	10^1	"ten to the first power"
(1)	1	"one"

In an expression as 10^2, the number 10 is called the base and the number 2 is called the exponent. The exponent tells how many times the base is taken as a factor in a product. 10^2 indicates (10×10) or 100. A number such as 10^2 is called a power of ten, and in this case it is the second power of ten. The exponent is sometimes omitted for the first power of ten; we usually write 10, instead of 10^1. All other exponents are always written. Another way to write $(4 \times 4 \times 4)$ is 4^3, where 4 is the base, and 3 is the exponent.

How can we write the meaning of "352" with exponents?

$$352 = (3 \times 10 \times 10) + (5 \times 10) + (2 \times 1)$$

$$= (3 \times 10^2) + (5 \times 10^1) + (2 \times 1).$$ This is called expanded notation. Writing numerals in expanded notation helps explain the meaning of the whole numeral.

History

Probably the reason that we use a numeral system with ten as a base is that people have ten fingers. This accounts for the fact that the ten symbols are called "digits" when they are used as numerals. We use the term "digits" when we wish to refer to the symbols apart from the longer numerals in which they are used.

The Celts, who lived in Europe more than 2000 years ago, used twenty as a base, and so did the Mayans in Central America. Can you think of a reason for this? What special name do we sometimes use for twenty? Some Eskimo tribes group and count by fives. Can you think of a good reason for this? Later we shall see how systems work when we use bases other than ten.

Exercises 2-3

1. Write the following in words: 10^1 as "ten to the first power," and so on up to 10^5.
2. Write each of the following using exponents.
   a. $3 \times 3 \times 3 \times 3 \times 3$
   b. $2 \times 2 \times 2 \times 2$
   c. $6 \times 6 \times 6 \times 6 \times 6 \times 6$
   d. $25 \times 25 \times 25$

e. $5 \times 5 \times 5 \times 5 \times 5 \times 5$
f. 4×4
g. $279 \times 279 \times 279 \times 279 \times 279$
h. 16

3. How many fives are used as factors in each of the following?
   a. 5^3
   b. 5^7
   c. 5^2
   d. 5^{10}
   e. 5^1
   f. 5^5

4. Write each of the following without exponents as $2^3 = 2 \times 2 \times 2$.
   a. 4^3
   b. 3^4
   c. 2^8
   d. 10^7
   e. 33^5
   f. 175^6

5. What does an exponent tell?

6. Write each of the following expressions as shown in the example: 4^3 means $4 \times 4 \times 4 = 64$.
   a. 3^3
   b. 5^2
   c. 4^4
   d. 2^5
   e. 6^2
   f. 7^3
   g. 8^2
   h. 9^2
   i. 10^3
   j. 3^4
   k. 2^6
   l. 4^5

7. Which numeral represents the larger number?
   a. 4^3 or 3^4
   b. 2^9 or 9^2

8. Write the following numerals in expanded notation as shown in the example: $210 = (2 \times 10^2) + (1 \times 10^1) + (0 \times 1)$
   a. 468
   b. 5324
   c. 7062
   d. 59,126
   e. 109,180

9. Complete the table shown below for the powers of ten, starting with 10^{10} and working down. The next expression will be 10^9, and so on. Write the numeral represented by each expression, and then write each numeral in words. Continue until you reach 10^1.

Power	Numeral	Words
10^{10}	10,000,000,000	ten billion
10^9	1,000,000,000	one billion
10^8	?	?

10. In the table of problem 9, what is the relation between the exponent of a power of 10 and the zeros when that number is written in decimal notation?

11. Write the following numerals with exponents:
    a. 1000 c. 1,000,000
    b. 100,000 d. one hundred million

12. A mathematician was talking to a group of arithmetic students one day. They talked about a large number which they decided to call a "googol." A googol is 1 followed by 100 zeros. Write this with exponents.

13. BRAINBUSTER. What is the meaning of 10^2? of 10^1? What do you think should be the meaning of 10^0?

2-4. Numerals in Base Seven

You have known and used decimal numerals for a long time, and you may think you understand all about them. Some of their characteristics, however, may have escaped your notice simply because the numerals are familiar to you. In this section you will study a system of notation with a different base. This will increase your understanding of decimal numerals.

Suppose we found people living on Mars with seven fingers. Instead of counting by tens, a Martian might count by sevens. Let us see how to write numerals in base seven notation. This time we plan to work with groups of seven. Look at the x's below and notice how they are grouped in sevens with some x's left over.

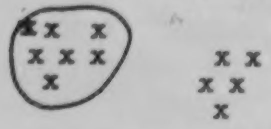


Figure 2-4-a

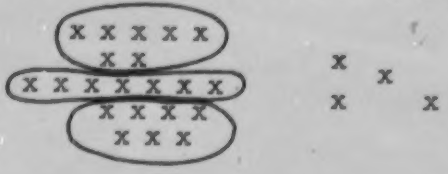


Figure 2-4-b

In figure 2-4-a we see one group of seven and five more. The numeral is written 15_{seven}. In this numeral, the 1 shows that there is one group of seven, and the 5 means that there are five ones.

In figure 2-4-b, how many groups of seven are there? How many x's are left outside the groups of seven? The numeral representing this number of x's is 34_{seven}. The 3 stands for three groups of seven, and the 4 represents four single x's or four ones. The "lowered" seven merely shows that we are working in base seven.

When we group in sevens the number of individual objects left can only be zero, one, two, three, four, five, or six. Symbols are needed to represent those numbers. Suppose we use the familiar 0, 1, 2, 3, 4, 5 and 6 for these rather than invent new symbols. As you will discover, no other symbols are needed for the base seven system.

If the x's are marks for days, we may think of 15_{seven} as a way of writing 1 week and five days. In our decimal system we name this number of days "twelve" and write it "12" to show one group of ten and two more. We do not write the base name in our numerals since we all know what the base is.

We should not use the name "fifteen" for 15_{seven} because fifteen is 1 ten and 5 more. We shall simply read 15_{seven} as "one, five, base seven."

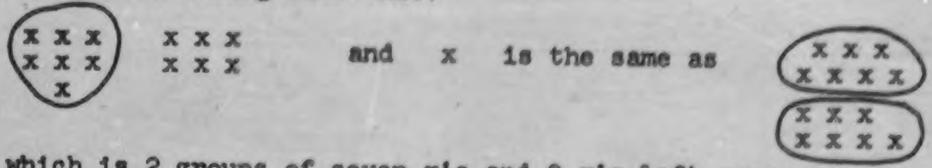
You know how to count in base ten and how to write the numerals in succession. Notice that one, two, three, four, five, six, seven, eight, and nine are represented by single symbols. How is the base number "ten" represented? This representation, 10, means one group of ten and zero more.

With this idea in mind, think about counting in base seven. Try it yourself and compare with the following table, filling in the numerals from 21_{seven} to 63_{seven}. In this table the "lowered" seven is omitted.

Counting in Base Seven

Number	Symbol	Number	Symbol
one	1	one, four	14
two	2	one, five	15
three	3	one, six	16
four	4	two, zero	20
five	5	two, one	21
six	6	-------	--
one, zero	10	six, three	63
one, one	11	six, four	64
one, two	12	six, five	65
one, three	13	six, six	66

How did you get the numeral following 16_{seven}? You probably thought something like this:



which is 2 groups of seven x's and 0 x's left over.

What would the next numeral after 66_{seven} be? Here you would have 6 sevens and 6 ones plus another one. This equals 6 sevens and another seven, that is, seven sevens. How could we represent $(\text{seven})^2$ without using a new symbol? We introduce a new group, the $(\text{seven})^2$ group. This number would then be written 100_{seven}. What does the number really mean? Go on from this point and write a few more numbers. What would be the next numeral after 666_{seven}?

Now you are ready to write a list of place values for base seven. Can you do this for yourself by studying the decimal place values on page 30 and thinking about the meaning of 100_{seven}?

Place Values in Base Seven

$(\text{seven})^5$	$(\text{seven})^4$	$(\text{seven})^3$	$(\text{seven})^2$	$(\text{seven})^1$	(one)

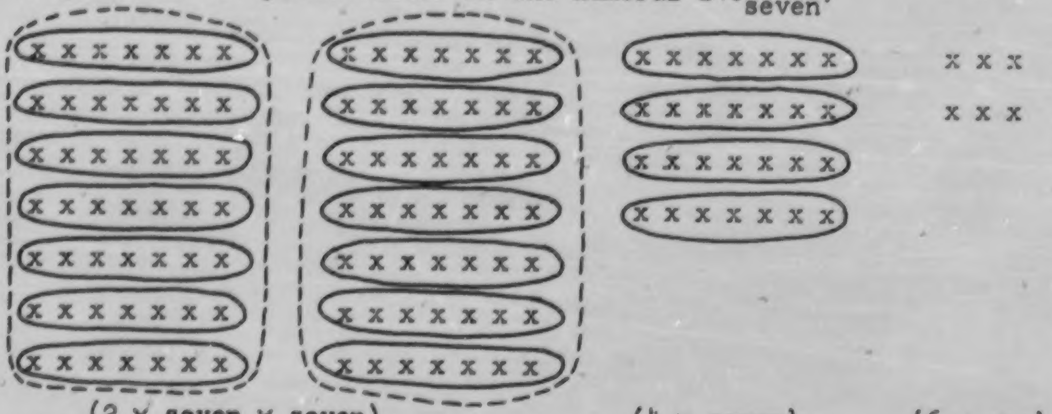
Notice that each place represents seven times the value of the next place to the right. The first place on the right is the one place in both the decimal and the seven systems. The value of the second place is the base times one. In this case what is it? The value in the third place from the right is (seven × seven), and in the next place (seven × seven × seven).

What is the decimal name for (seven × seven)? We need to use this (forty-nine) when we change from base seven to base ten. Show that the decimal numeral for $(\text{seven})^3$ is 343. What is the decimal numeral for $(\text{seven})^4$?

Using the chart above, we see that

$246_{\text{seven}} = (2 \times \text{seven} \times \text{seven}) + (4 \times \text{seven}) + (6 \times \text{one})$.

The diagram shows the actual grouping represented by the digits and the place values in the numeral 246_{seven}:



If we wish to write the number of x's above in the decimal system of notation we may write:

$$\begin{aligned} 246_{\text{seven}} &= (2 \times 7 \times 7) + (4 \times 7) + (6 \times 1) \\ &= (2 \times 49) \quad + (4 \times 7) + (6 \times 1) \\ &= \quad 98 \quad + \quad 28 \quad + \quad 6 \\ &= 132_{\text{ten}} \end{aligned}$$

Regroup the x's above to show that there are 1 (ten × ten) group, 3(ten) groups, and 2 more. This should help you understand that $246_{\text{seven}} = 132_{\text{ten}}$.

Exercises 2-4

1. Group the x's below and write the number of x's in base seven notation:

   a. x x x x x

   x x x x x

   b. x x x x x

   x x x x x

   x x x x

   x x x x

   c. x x x x x x x x x
   x x x x x x x x x
   x x x x x x x x x
   x x x x x x x x x
   x x x x x x x x x
   x x x x x x x x x
   x x x x x x x x

2. Draw x's and group them to show the meaning of the following numerals.

   a. 11_{seven} b. 26_{seven} c. 35_{seven} d. 101_{seven}.

3. Write each of the following numerals in expanded form and then in decimal notation.

   a. 33_{seven} b. 45_{seven} c. 100_{seven} d. 524_{seven}.

4. Write the next consecutive numeral after each of the following numerals.

   a. 6_{seven} c. 54_{seven} e. 666_{seven}
   b. 10_{seven} d. 162_{seven} f. 1006_{seven}.

5. What is the value of the "6" in each of the following numerals?

   a. 560_{seven} b. 56_{seven} c. 605_{seven} d. 6050_{seven}

6. In the base seven system write the value of the fifth place counting left from the units place.

7. In the base seven system, what is the value of the tenth place from the right?

*8. What numeral in the seven system represents the number named by six dozen?

9. Which number is larger, 452_{seven} or 432_{seven}?

10. Which number is greater, 250_{seven} or 205_{ten}?

11. Which is smaller, 2125_{seven} or 754_{ten}?

12. A number is divisible by ten if a remainder of zero is obtained when the number is divided by ten.

a. Is 30_{ten} divisible by ten? Why?

b. Is 241_{ten} divisible by ten? Why?

c. How can you tell by glancing at a base ten numeral whether the number is divisible by ten?

*13. Is 30_{seven} divisible by ten? Explain how you arrived at your answer. Is 60_{seven} divisible by ten?

*14. a. What would the phrase "a counting number is divisible by seven" mean?

b. Is 30_{seven} divisible by seven? Why?

*15. Is 31_{seven} divisible by seven? Explain your answer.

*16. State a rule for determining when a number written in base seven is divisible by seven.

You should see from problems 11-15 that the way we determine whether a number is divisible by ten depends on the system in which it is written. The rule for divisibility by ten in the decimal system is similar to the rule for divisibility by seven in the base seven system.

17. Which of the numbers 24_{ten}, 31_{ten}, and 68_{ten} are divisible by two? How do you tell? What do you call a number which is divisible by two? What do you call a number not divisible by two?

*18. Is 11_{seven} an even number or an odd number? Can you tell simply by glancing at the following which represent even or odd numbers?

12_{seven}, 13_{seven}, 14_{seven}, 25_{seven}, 66_{seven}.

What could you do to tell?

Here again a rule for divisibility in base ten will not work for base seven. Rules for divisibility seem to depend on the base with which we are working.

19. BRAINBUSTER. On planet X-101 the pages in books are numbered in order as follows: 1, ∠, △, □, ⧄, ⊠, 1-, 11, 1∠, 1△, 1□, 1⧄, 1⊠; ∠-, ∠1, and so forth. What seems to be the base of the numeration system these people use? Why? How would the next number after ∠1 be written? Which

symbol corresponds to our zero? Write numerals for numbers from □- to ⧅Δ.

20. BRAINBUSTER. Find a rule for determining when a number written in base seven is divisible by two.

2-5. Computation in Base Seven

Addition

In the decimal, or base ten, system there are 100 "basic" addition combinations. By this time you know all of them. The combinations can be arranged in a convenient table. Part of the table is given below:

Addition, Base Ten

+	0	1	2	3	4	5	6	7	8	9
0	0									
1	1	2								
2				5						
3	3	4	5	6						
4	4	5	6	7	8					
5	5	6	7	8	9	10	11			
6										
7										
8										
9									17	

The numbers represented in the horizontal row above the line at the top of the table are added to the numbers in the vertical row under the "+" sign at the left. The sum of each pair of numbers is written in the table. The sum 2 + 3 is 5, as pointed out by the arrows.

[sec. 2-5]

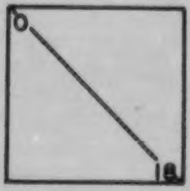
Exercises 2-5-a

1. Find the sums

   a. 6 + 5 b. 9 + 8.

2. Use cross ruled paper and complete the addition table on page 39 (you will use it later).

3. Draw a diagonal line from the upper left corner to the lower right corner of the chart as shown at the right.



   a. Is 3 + 4 the same as 4 + 3?

   b. How could the answer to part a) be determined from the chart?

   c. What do you notice about the two parts of the chart?

   d. What does this tell you about the number of different combinations which must be mastered? Be sure you can recall any of these combinations whenever you need them.

4. Make a chart to show the basic sums when the numbers are written in base seven notation. Four sums are supplied to help you.

+	0	1	2	3	4	5	6
0							
1				4			
2					6		
3						11	
4							13
5							
6							

5. a. How many different number combinations are there in the base seven table? Why?

   b. Which would be easier, to learn the necessary multiplication combinations in base seven or in base ten? Why?

c. Find $4_{ten} + 5_{ten}$ and $4_{seven} + 5_{seven}$ from the tables. Are the results equal; that is, do they represent the same number?

The answer to problem 5c is an illustration of the fact that a number is an idea independent of the numerals used to write its name. Actually, 9_{ten} and 12_{seven} are two different names for the same number.

Do not try to memorize the addition combinations for base seven. The value in making the table lies in the help it gives you in understanding operations with numbers.

The table that you completed in problem 4 of the last set of exercises shows the sums of pairs of numbers from zero to six. Actually, little more is needed to enable us to add larger numbers. In order to see what else in needed, let us consider how we add in base ten. What are the steps in your thinking when you add numbers like twenty-five and forty-eight in the decimal notation?

$$\begin{array}{lcl}
25 & = & 2 \text{ tens} + 5 \text{ ones} & = & \xrightarrow{\qquad\qquad\qquad} & 25 \\
\underline{48} & = & \underline{4 \text{ tens} + 8 \text{ ones}} & = & \xrightarrow{\qquad\qquad\qquad} & \underline{48} \\
 & & 6 \text{ tens} + 13 \text{ ones} & = & 7 \text{ tens} + 3 \text{ ones} = & 73
\end{array}$$

Try adding in base seven: $14_{seven} + 35_{seven}$

$$\begin{array}{l}
1 \text{ seven} + 4 \text{ ones} \\
\underline{3 \text{ sevens} + 5 \text{ ones}} \\
4 \text{ sevens} + 12 \text{ ones} = 5 \text{ sevens} + 2 \text{ ones} = 52_{seven}.
\end{array}$$

(You may look up the sums 5 + 4 and 3 + 1 in the base seven addition table.

How are the two examples alike? How are they different? When is it necessary to "carry" (or regroup) in the ten system? When is it necessary to "carry" (or regroup) in the seven system?

Try your skill in addition on the following problems. Use the addition table for the basic sums.

$$\begin{array}{r} 42_{seven} \\ \underline{13}_{seven} \end{array} \quad
\begin{array}{r} 65_{seven} \\ \underline{11}_{seven} \end{array} \quad
\begin{array}{r} 32_{seven} \\ \underline{25}_{seven} \end{array} \quad
\begin{array}{r} 254_{seven} \\ \underline{105}_{seven} \end{array} \quad
\begin{array}{r} 435_{seven} \\ \underline{625}_{seven} \end{array} \quad
\begin{array}{r} 524_{seven} \\ \underline{564}_{seven} \end{array}$$

The answers in order are 55_{seven}, 106_{seven}, 60_{seven}, 362_{seven}, 1363_{seven}, and 1421_{seven}.

Subtraction

How did you learn to subtract in base ten? You probably used subtraction combinations such as 14 - 5 until you were thoroughly familiar with them. You know the answer to this problem but suppose, for the moment, that you did not. Could you get the answer from the addition table? You really want to ask the following question "What is the number which, when added to 5, yields 14?" Since the seventh row of the base ten addition table gives the results of adding various numbers to 5, we should look for 14 in that row. Where do you find the answer to 14 - 5? Did you answer "the last column"? Use the base ten addition table to find

9 - 2, 8 - 5, 12 - 7, 17 - 9.

The idea discussed above is used in every subtraction problem. One other idea is needed in many problems, the idea of "borrowing" or "regrouping." This last idea is illustrated below for base ten to find 761 - 283:

$$\begin{array}{lcrcr}
7\text{ hundreds} + 6\text{ tens} + 1\text{ one} & = & 6\text{ hundreds} + 15\text{ tens} + 11\text{ ones} & = & 761 \\
\underline{2\text{ hundreds} + 8\text{ tens} + 3\text{ ones}} & = & \underline{2\text{ hundreds} + 8\text{ tens} + 3\text{ ones}} & = & \underline{283} \\
 & & 4\text{ hundreds} + 7\text{ tens} + 8\text{ ones} & = & 478
\end{array}$$

Now let us try subtraction in base seven. How would you find $6_{\text{seven}} - 2_{\text{seven}}$? Find $13_{\text{seven}} - 6_{\text{seven}}$. How did you use the addition table for base seven? Find answers to the following subtraction examples:

$$\begin{array}{ccccc}
15_{\text{seven}} & 12_{\text{seven}} & 11_{\text{seven}} & 14_{\text{seven}} & 13_{\text{seven}} \\
\underline{6_{\text{seven}}} & \underline{4_{\text{seven}}} & \underline{6_{\text{seven}}} & \underline{5_{\text{seven}}} & \underline{4_{\text{seven}}}
\end{array}$$

The answers to these problems are 6_{seven}, 5_{seven}, 2_{seven}, 6_{seven}, and 6_{seven}.

Let us work a harder subtraction problem in base seven comparing the procedure with that used above:

$$\begin{array}{lclcrcr}
43_{\text{seven}} & = & 4\text{ sevens} + 3\text{ ones} & = & 3\text{ sevens} + 13\text{ ones} & = & 43_{\text{seven}} \\
\underline{16_{\text{seven}}} & = & \underline{1\text{ seven} + 6\text{ ones}} & = & \underline{1\text{ seven} + 6\text{ ones}} & = & \underline{16_{\text{seven}}} \\
 & & & & 2\text{ sevens} + 4\text{ ones} & = & 24_{\text{seven}}
\end{array}$$

Be sure to note that "13 ones" above is in the seven system and is "one seven, three ones." If you wish to find the number you add to 6_{seven} to get 13_{seven}, how can you use the table to help you? Some of you may think of the number without referring to the table.

Practice on these subtraction examples:

$$\begin{array}{r} 56_{seven} \\ \underline{14_{seven}} \end{array} \qquad \begin{array}{r} 61_{seven} \\ \underline{35_{seven}} \end{array} \qquad \begin{array}{r} 34_{seven} \\ \underline{26_{seven}} \end{array} \qquad \begin{array}{r} 452_{seven} \\ \underline{263_{seven}} \end{array} \qquad \begin{array}{r} 503_{seven} \\ \underline{140_{seven}} \end{array}$$

The answers are 42_{seven}, 23_{seven}, 5_{seven}, 156_{seven} and 333_{seven}.

Exercises 2-5-b

1. Each of the following examples is written in base seven. Add. Check by changing the numerals to decimal notation and adding in base ten as in the example:

$$\begin{array}{rcr} \text{Base Seven} & & \text{Base Ten} \\ 16_{seven} & = & 13 \\ 23_{seven} & = & 17 \\ \hline 42_{seven} & & 30 \end{array}$$

Does $42_{seven} = 30$?

a. $\begin{array}{r} 25_{seven} \\ \underline{31_{seven}} \end{array}$ b. $\begin{array}{r} 56_{seven} \\ \underline{21_{seven}} \end{array}$ c. $\begin{array}{r} 214_{seven} \\ \underline{53_{seven}} \end{array}$

d. $160_{seven} + 430_{seven}$ e. $45_{seven} + 163_{seven}$

f. $403_{seven} + 563_{seven}$ g. $645_{seven} + 605_{seven}$

h. $6245_{seven} + 5314_{seven}$ i. $6204_{seven} + 234_{seven}$

j. $645_{seven} + 666_{seven}$ k. $5406_{seven} + 6245_{seven}$.

2. Use the base seven addition table to find:

a. $6_{seven} - 4_{seven}$ b. $11_{seven} - 4_{seven}$

c. $12_{seven} - 5_{seven}$

3. Each of the following examples is written in base seven. Subtract. Check by changing to decimal numerals.

a. $\begin{array}{r} 10_{seven} \\ \underline{5_{seven}} \end{array}$ b. $\begin{array}{r} 65_{seven} \\ \underline{26_{seven}} \end{array}$ c. $\begin{array}{r} 200_{seven} \\ \underline{4_{seven}} \end{array}$ d. $\begin{array}{r} 160_{seven} \\ \underline{6_{seven}} \end{array}$

e. $44_{seven} - 35_{seven}$ f. $641_{seven} - 132_{seven}$

g. $502_{seven} - 266_{seven}$ h. $5000_{seven} - 4261_{seven}$

i. $634_{seven} - 52_{seven}$ j. $134_{seven} - 65_{seven}$

k. $3451_{seven} - 2164_{seven}$ l. $253_{seven} - 166_{seven}$

4. Show by grouping x's that:

a. 4 twos = 11_{seven} c. 3 fives = 21_{seven}

b. 6 threes = 24_{seven} d. 5 sixes = 42_{seven}

Multiplication

In order to multiply, we may use a table of basic facts. Complete the following table in decimal numerals and be sure you know and can recall instantly the product of any two numbers from zero to nine.

Multiplication, Base Ten

×	0	1	2	3	4	5	6	7	8	9
0	0	0								
1	0	1								
2			4	6						
3				9	12					
4										
5										
6										
7										
8										
9										

Exercises 2-5-c

1. Refer to the preceding table.
   a. Explain the row of zeros and the column of zeros.
   b. Which row in the table is exactly like the row at the top? Why?
2. Imagine a diagonal line drawn from the × sign in the table to the lower right corner. What can you say about the two triangular parts of the table on each side of the line?
3. Complete the multiplication table below for base seven. Suggestion: To find $4_{seven} \times 3_{seven}$ you could write four x's three times and regroup to show the base seven numeral. Better still, you might think of this as $3_{seven} + 3_{seven} + 3_{seven} + 3_{seven}$.

Multiplication. Base Seven

×	0	1	2	3	4	5	6
0							
1							
2					11	13	
3							
4				15			
5							
6							51

4. There are fewer entries in the base seven table than in the table for base ten. What does this fact tell you about the ease of learning multiplication in base seven?
5. Imagine the diagonal line drawn from the "×" sign to the lower right-hand corner of the last table.
   a. How are the entries above the diagonal line related to those below it?
   b. What fact does the observation of part a tell you about $3_{seven} \times 4_{seven}$?

There is no value in memorizing the table for base seven. The value of this table lies in your understanding of it.

6. Multiply the following numbers in base ten numerals:

$$\begin{array}{r} 45 \\ \times\ 63 \\ \hline \end{array} \qquad \begin{array}{r} 249 \\ \times\ 75 \\ \hline \end{array} \qquad \begin{array}{r} 4627 \\ \times\ 436 \\ \hline \end{array} \qquad \begin{array}{r} 7834 \\ \times\ 89 \\ \hline \end{array} \qquad \begin{array}{r} 51043 \\ \times\ 78 \\ \hline \end{array}$$

You know about carrying (or regrouping) in addition, and you have had experience in multiplication in base ten. Use the base seven multiplication table to find the following products.

$$\begin{array}{r} 52_{seven} \\ \times\ 3_{seven} \\ \hline \end{array} \qquad \begin{array}{r} 34_{seven} \\ \times\ 6_{seven} \\ \hline \end{array} \qquad \begin{array}{r} 421_{seven} \\ \times\ 4_{seven} \\ \hline \end{array} \qquad \begin{array}{r} 621_{seven} \\ \times\ 2_{seven} \\ \hline \end{array} \qquad \begin{array}{r} 604_{seven} \\ \times\ 35_{seven} \\ \hline \end{array}$$

The answers are 216_{seven}, 303_{seven}, 2314_{seven}, 1542_{seven}, 31406_{seven}

Check the multiplication shown at the right and then answer the following questions. How do you get the entry 123 on the third line? How do you get the entry 201 on the fourth line? Why is the 1 on line 4 placed under the 2 on line 3? Why is the 0 on line 4 placed under the 1 on line 3? If you do not know why the entries on lines 3 and 4 are added to get the answer, you will study this more thoroughly later.

$$\begin{array}{r} 45_{seven} \\ \times\ 32_{seven} \\ \hline 123\phantom{_{seven}} \\ 201\phantom{0_{seven}} \\ \hline 2133_{seven} \end{array}$$

One way to check your work is to change the base seven numerals to base ten numerals as shown here:

$$\begin{array}{rlr} 604_{seven} & = (6 \times 49) + (0 \times 7) + (4) = 294 + 4 = & 298_{ten} \\ \times\ 35_{seven} & = (3 \times 7) + (5) \qquad\qquad\quad = 21 + 5 = & \times\ 26_{ten} \\ \hline 4226\phantom{_{seven}} & & 1788\phantom{_{ten}} \\ 2415\phantom{0_{seven}} & & 596\phantom{0_{ten}} \\ \hline 31406_{seven} & = (3 \times 2401) + (1 \times 343) + (4 \times 49) + (0 \times 7) + (6) = & 7748_{ten} \end{array}$$

Division

Division is left as an exercise for you. You may find that it is not easy. Working in base seven should help you understand why some boys and girls have trouble with division in base ten. Here are two examples you may wish to examine. All the numerals within the examples are written in base seven. How can you use the multiplication table here?

Division in Base Seven

$$\begin{array}{r@{}l}
 & 454_{\text{seven}} \\
6_{\text{seven}}\ ) & \overline{4053}_{\text{seven}} \\
 & \underline{33} \\
 & 45 \\
 & \underline{42} \\
 & 33 \\
 & \underline{33}
\end{array}
\qquad\qquad
\begin{array}{r@{}l}
 & 2015_{\text{seven}} \\
46_{\text{seven}}\ ) & \overline{126125}_{\text{seven}} \\
 & \underline{125} \\
 & 112 \\
 & \underline{46} \\
 & 335 \\
 & \underline{332} \\
 & 3
\end{array}$$

Exercises 2-5-d

1. Multiply the following numbers in base seven numerals and check your results in base 10.

   a. $14_{\text{seven}} \times 3_{\text{seven}}$ b. $6_{\text{seven}} \times 25_{\text{seven}}$

   c. $63_{\text{seven}} \times 12_{\text{seven}}$ d. $5_{\text{seven}} \times 461_{\text{seven}}$

   e. $56_{\text{seven}} \times 43_{\text{seven}}$ f. $654_{\text{seven}} \times 453_{\text{seven}}$

   g. $3046_{\text{seven}} \times 24_{\text{seven}}$ h. $5643_{\text{seven}} \times 652_{\text{seven}}$

   i. $250_{\text{seven}} \times 341_{\text{seven}}$ j. $26403_{\text{seven}} \times 45_{\text{seven}}$

*2. Divide. All numerals in this exercise are in base seven.

   a. $6_{\text{seven}}\overline{)\,42_{\text{seven}}}$ b. $5_{\text{seven}}\overline{)\,433_{\text{seven}}}$

   c. $4_{\text{seven}}\overline{)\,2316_{\text{seven}}}$ d. $21_{\text{seven}}\overline{)\,2625_{\text{seven}}}$

3. Write in expanded form:

   a. 403_{seven} b. 189_{ten}

4. Which of the numerals in Exercise 3 represents the larger number?

5. Add the following:

   a. $52_{\text{seven}} + 14_{\text{seven}}$ b. $\begin{array}{r} 65_{\text{seven}} \\ \underline{25_{\text{seven}}} \end{array}$

[sec. 2-5]

c. $$\begin{array}{r} 434_{seven} \\ \underline{324_{seven}} \end{array}$$

d. $601_{seven} + 304_{seven}$

6. Subtract the following:

a. $$\begin{array}{r} 13_{seven} \\ \underline{6_{seven}} \end{array}$$

b. $30_{seven} - 1_{seven}$

c. $$\begin{array}{r} 402_{seven} \\ \underline{35_{seven}} \end{array}$$

7. Rewrite the following paragraph replacing the base seven numerals with base ten numerals.

Louise takes grade 10_{seven} mathematics in room 234_{seven}. The book she uses is called *Junior High School Mathematics* $\underline{10}_{seven}$. It has 21_{seven} chapters and 1102_{seven} pages. There are 44_{seven} pupils in the class which meets 5_{seven} times each week for 106_{seven} minutes daily. 16_{seven} of the pupils are girls and 25_{seven} are boys. The youngest pupil in the class is 14_{seven} years old and the tallest is 123_{seven} inches tall.

2-6. Changing from Base Ten to Base Seven

You have learned how to change a number written in base seven numerals to base ten numerals. It is also easy to change from base ten to base seven. Let us see how this is done.

In base seven, the values of the places are: one, $seven^1$, $seven^2$, $seven^3$, and so on. That is, the place values are one and the powers of seven.

$$seven^1 = 7_{ten}$$

$$seven^2 = (7 \times 7) \text{ or } 49_{ten}$$

$$seven^3 = (7 \times 7 \times 7) \text{ or } 343_{ten}$$

Suppose you wished to change 12_{ten} to base seven numerals. This time we shall *think* of groups of powers of seven instead of actually grouping marks. What is the largest power of seven which is contained in 12_{ten}? Is $seven^1$ the largest? How about $seven^2$ (forty-nine) or $seven^3$ (three hundred forty-three)?

We can see that only seven^1 is small enough to be contained in 12_{ten}.

When we divide 12 by 7 we have

$$\begin{array}{r} 1 \\ 7\overline{)12} \\ 7 \\ \hline 5. \end{array}$$

What does the 1 on top mean? What does the 5 mean? They tell us that 12_{ten} contains 1 seven with 5 units left over, or that $12_{\text{ten}} = (1 \times \text{seven}) + (5 \times \text{one})$. Thus $12_{\text{ten}} = 15_{\text{seven}}$.

Be sure you know which place in a base seven numeral has the value seven^2, the value seven^3, the value seven^4, and so on.

How is 54_{ten} regrouped for base seven numerals? What is the largest power of seven which is contained in 54_{ten}?

In 54_{ten} we have $\underline{\ \ ?\ \ } \times \text{seven}^2 + \underline{\ \ ?\ \ } \times \text{seven} + \underline{\ \ ?\ \ } \times \text{one}$.

$$\begin{array}{r} 1 \\ 49\overline{)54} \\ 49 \\ \hline 5 \end{array}$$

We have $(\underline{\ \ 1\ \ } \times \text{seven}^2) + (0 \times \text{seven}) + (\underline{\ \ 5\ \ } \times \text{one})$.
Then $54_{\text{ten}} = 105_{\text{seven}}$.

Suppose the problem is to change 524_{ten} to base seven numerals. Since 524_{ten} is larger than 343 (seven^3), find how many 343's there are.

$$\begin{array}{r} 1 \\ 343\overline{)524} \\ 343 \\ \hline 181 \end{array}$$

Thus 524 contains one seven^3 with 181 remaining, or $524 = (1 \times \text{seven}^3) + 181$, and there will be a "1" in the seven^3 place.

Now find how many 49's (seven^2) there are in the remaining 181

$$\begin{array}{r} 3 \\ 49\overline{)181} \\ 147 \\ \hline 34 \end{array}$$

Thus 181 contains 3 49's with 34 remaining, or $181 = (3 \times \text{seven}^2) + 34$, and there will be a "3" in the seven^2 place.

How many sevens are there in the remaining 34?

$$\begin{array}{r} 4 \\ 7\,\overline{)\,34} \\ \underline{28} \\ 6 \end{array}$$

Thus 34 contains 4 seven's with 6 remaining, or 34 = (4 × seven) + 6, and there will be a "4" in the sevens place.

What will be in the units place? We have:

$524_{ten} = (1 \times seven^3) + (3 \times seven^2) + (4 \times seven) + (6 \times one)$

$524_{ten} = 1346_{seven}$.

Cover the answers below until you have made the changes for yourself.

$10_{ten} = (1 \times seven) + (3 \times one) = 13_{seven}$

$46_{ten} = (6 \times seven) + (4 \times one) = 64_{seven}$

$162_{ten} = (3 \times seven^2) + (2 \times seven) + (1 \times one) = 321_{seven}$

$1738_{ten} = (5 \times seven^3) + (0 \times seven^2) + (3 \times seven) + (2 \times one)$
$= 5032_{seven}$.

In changing base ten numerals to base seven we first select the largest place value of base seven (that is, power of seven) contained in the number. We divide the number by this power of seven and find the quotient and remainder. The quotient is the first digit in the base seven numeral. We divide the remainder by the next smaller power of seven and this quotient is the second digit. We continue to divide remainders by each succeeding, smaller power of seven to determine all the remaining digits in the base seven numeral.

Exercises 2-6

1. Show that:

   a. $50_{ten} = 101_{seven}$

   b. $145_{ten} = 265_{seven}$

   c. $1024_{ten} = 2662_{seven}$

2. Change the following base ten numerals to base seven numerals:

a.	12	d.	53
b.	36	e.	218
c.	44	f.	1320

Problems 3, 4, and 5 will help you discover another method for changing base ten numerals to base seven.

3. Divide 1958_{ten} by ten. What is the quotient? What is the remainder? Divide the quotient by ten. What is the new quotient? The new remainder? Continue in the same way, dividing each quotient by ten until you get a quotient of zero. How are the successive remainders related to the original number? Try the same process with $123{,}456{,}789_{ten}$. Try it with any other number.
4. Divide 524_{ten} by seven. What is the quotient? The remainder? Divide the quotient by seven and continue as in Exercise 3, except that this time divide by seven instead of ten. Now write 524_{ten} as a base seven numeral and compare this with the remainders which you have obtained.
5. Can you now discover another method for changing from base ten to base seven numerals?
6. In each of the examples below there are some missing numerals. Supply the numerals which will make the examples correct. Remember that if no base name is given, then the base is ten.

a. Addition:
$$\begin{array}{r} 675 \\ 486 \\ \hline ???? \end{array}$$

b. Addition:
$$\begin{array}{r} 8\ 9\ 4 \\ ?\ ?\ ? \\ \hline 1\ 1\ 6\ 9 \end{array}$$

c. Addition:
$$\begin{array}{r} 4\ 3\ 2_{seven} \\ ?\ ?\ ?_{seven} \\ \hline 1\ 4\ 1\ 6_{seven} \end{array}$$

d. Addition:
$$\begin{array}{r} 2\ 3\ 0\ 5_{seven} \\ ?\ ?\ ?_{seven} \\ \hline 3\ 1\ 0\ 0_{seven} \end{array}$$

e. Addition:

$$\begin{array}{r} 2\,6\,4_{seven} \\ 3\,5\,2_{seven} \\ 1\,4\,0_{seven} \\ \hline \end{array}$$

f. Multiplication:

$$\begin{array}{r} 5\,1\,4_{seven} \\ \times \quad ?_{seven} \\ \hline 2\,1\,4\,5_{seven} \end{array}$$

*g. Multiplication:

$$\begin{array}{r} ?\,?\,?_{seven} \\ \times \; 5\,4_{seven} \\ \hline 3\,6\,2\,0\,1_{seven} \end{array}$$

2-7. Numerals in Other Bases

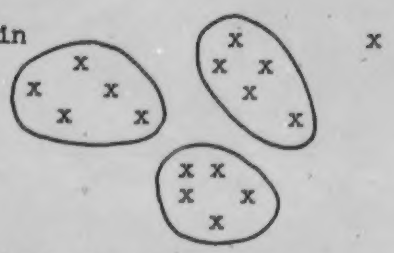
You have studied base seven numerals, so you now know that it is possible to express numbers in systems different from the decimal scale. Many persons think that the decimal system is used because the base ten is superior to other bases, or because the number ten has special properties. Earlier it was indicated that we probably use ten as a base because man has ten fingers. It was only natural for primitive people to count by making comparisons with their fingers. If man had had six or eight fingers, he might have learned to count by sixes or eights.

Our familiar decimal system of notation is superior to the Egyptian, Babylonian, and others because it uses the idea of place value and has a zero symbol, not because its base is ten. The Egyptian system was a tens system, but it lacked efficiency for other reasons.

Bases Five and Six

Our decimal system uses ten symbols. In the seven system you used only seven symbols, 0, 1, 2, 3, 4, 5, and 6. How many symbols would Eskimos use counting in base five? How many symbols would base six require? A little thought on the preceding questions should lead you to the correct answers. Can you suggest how many symbols are needed for base twenty?

The x's at the right are grouped in sets of five. How many groups of five are there? How many ones are left?



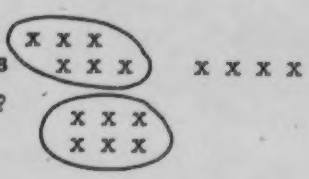
The decimal numeral for the number of x's in this diagram is 16. Using the symbols 0, 1, 2, 3, and 4, how would 16_{ten} be represented in base five numerals? An Eskimo, counting in base five, would think:

there are 3 groups of five and 1 more,

$$16_{ten} = (3 \times \text{five}) + (1 \times \text{one}),$$

$$16_{ten} = 31_{five}.$$

In the drawing at the right sixteen x's are grouped by sixes. How many groups of six are there? Are there any x's left? How would you write 16_{ten} in base six numerals?



There are 2 groups of six and 4 more,

$$16_{ten} = (2 \times \text{six}) + (4 \times \text{one}),$$

$$16_{ten} = 24_{six}.$$

Write sixteen x's. Enclose them in groups of four x's. Can you write the numeral 16_{ten} in base four numerals? How many groups of four are there? Remember, you cannot use the symbol "4" in base four. A table of the powers of four in decimal numerals is shown below.

(four^3)	(four^2)	(four^1)	(one)
$(4 \times 4 \times 4)$	(4×4)	(4)	(1)
(64)	(16)	(4)	(1)

To write sixteen x's in base four we need $(1 \text{ group of four}^2) + (0 \text{ groups of four}) + (0 \text{ ones})$. That is,

$16_{\text{ten}} = 100_{\text{four}}$.

Exercises 2-7

1. Draw sixteen x's. Group the x's in sets of three.
   a. There are _____ groups of three and _____ left over.
   b. Are your answers to part (a) both digits in the base three system? Why not?
   c. In sixteen x's there are (_____ groups of three^2) + (_____ groups of three) + (_____ left over).
   d. $16_{\text{ten}} = \_\_\_\_\__{\text{three}}$.
2. Draw groups of x's to show the numbers represented by the following numerals. Then write the decimal numerals for these numbers.
   a. 23_{four} b. 15_{six} c. 102_{three} d. 21_{five}.
3. Write in base five notation the numbers from one through thirty. Start a table as shown below:

Base ten	0	1	2	3	4	5	6	7
Base five	0	1	?	?	?	?	?	?

4. a. How many threes are there in 20_{three}?
   b. How many fours are there in 20_{four}?
   c. How many fives are there in 20_{five}?
   d. How many sixes are there in 20_{six}?
5. Write the following in expanded notation. Then write the base ten numeral for each as shown in the example.
   Example: $102_{\text{five}} = (1 \times 25) + (0 \times 5) + (2 \times 1) = 27$
   a. 245_{six} c. 1002_{three}
   b. 412_{five} d. 1021_{four}
6. Write the following decimal numerals in bases six, five, four and three. Remember the values of the powers for each of these bases. Note the example:

$$7_{\text{ten}} = 11_{\text{six}} = 12_{\text{five}} = 13_{\text{four}} = 21_{\text{three}}.$$

   a. 11_{ten} b. 15_{ten}

c. 28_{ten} d. 36_{ten}

7. What is the smallest whole number which can be used as a base for a system of notation?

*8. Do the following computations:

a. Add: $132_{four} + 211_{four}$.

b. Add: $15_{six} + 231_{six} + 420_{six}$.

c. Subtract: $1211_{three} - 202_{three}$.

d. Subtract: $1423_{five} - 444_{five}$.

e. Multiply: $13_{four} \times 3_{four}$.

f. Multiply: $121_{six} \times 5_{six}$.

g. $4_{six}\overline{)\,452_{six}}$

h. $2_{three}\overline{)\,121_{three}}$

9. BRAINBUSTER. Make up a place value system where the following symbols are used:

Symbol	Decimal Value	Name
0	0	do
1	1	re
∧	2	mi
≥	3	fa
10	4	re do

Write the numerals for numbers from zero to twenty in this system. Write the names in words using "do, re," etc.

10. BRAINBUSTER. Using the symbols and scale from the first Brainbuster, complete the addition and multiplication tables shown below.

+	0	1	∧	≥
0				
1				
∧				
≥				

×	0	1	∧	≥
0				
1				
∧				
≥				

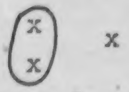
2-8. The Binary and Duodecimal Systems

There are two other bases of special interest. The base two, or binary, system is used by some modern, high speed computing machines. These computers, sometimes incorrectly called "electronic brains," use the base two as we use base ten. The twelve, or duodecimal, system is considered by some people to be a better base for a system of notation than ten.

Binary System

Historians tell of primitive people who used the binary system. Some Australian tribes still count by pairs, "one, two, two and one, two twos, two twos and one," and so on.

The binary system groups by pairs as is done with the three x's at the right. How many groups of two are shown? How many single x's are left? Three x's means 1 group of two and 1 one. In binary notation the numeral 3_{ten} is written 11_{two}.



Counting in the binary system starts as follows:

Decimal numerals	1	2	3	4	5	6	7	8	9	10
Binary numerals	1	10	11	100	101	110	111	?	?	?

How many symbols are needed for base two numerals? Notice that the numeral 101_{two} represents the number of fingers on one hand. What does 111_{two} mean?

$$111_{two} = (1 \times two^2) + (1 \times two^1) + (1 \times one) = 4 + 2 + 1 = 7_{ten}.$$

How would you write 8_{ten} in binary notation? How would you write 10_{ten} in binary notation? Compare this numeral with 101_{two}.

Modern high speed computers are electrically operated. A simple electric switch has only two positions, open (on) or closed (off). Computers operate on this principle. Because there are only two positions for each place, the computers use the binary system of notation.

We will use the drawing at the right to represent a computer. The four circles represent four lights on a panel, and each light represents one place in the binary system. When the current is flowing the light is on, shown in Figure 2-8b as ● . A ● is represented by the symbol "1". When the current does not flow, the light is off, shown by ○ in Figure 2-8b. This is represented by the symbol "0". The panel in Figure 2-8b represents the binary numeral 1010_{two}. What decimal numeral is represented by this numeral? The table at the right shows the place values for the first five places in base two numerals.

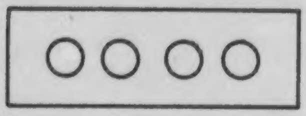


Figure 2-8a

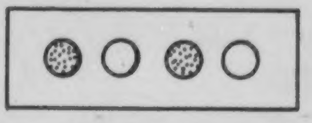


Figure 2-8b

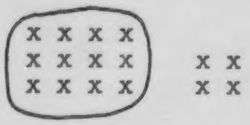
two^4	two^3	two^2	two^1	one
2x2x2x2	2x2x2	2x2	2	1
16	8	4	2	1

$$\begin{aligned} 1010_{\text{two}} &= (1 \times \text{two}^3) + (0 \times \text{two}^2) + (1 \times \text{two}^1) + (0 \times \text{one}) \\ &= (1 \times 8) + (0 \times 4) + (1 \times 2) + (0 \times 1) \\ &= 10_{\text{ten}}. \end{aligned}$$

Duodecimal System

In the twelve, or duodecimal, system, we group by twelves. We frequently count in dozens, as with a dozen eggs, a dozen rolls, or a dozen pencils. Twelve dozen (12 × 12) is called a gross. Schools sometimes buy pencils by the gross.

The sixteen x's shown at the right are grouped as one group of twelve with four x's left. Written as a base twelve numeral,



$$16_{\text{ten}} = (1 \times \text{twelve}) + (4 \times \text{one}) = 14 \text{ twelve}.$$

Draw twenty-five x's on a sheet of paper. Draw circles around groups of twelve. How many groups of twelve are there?

Are any x's left over? How would you write 25_{ten} in duodecimal notation? Can you see why it is written 21_{twelve}?

$$25_{ten} = (2 \times \text{twelve}) + (1 \times \text{one}) = 21_{twelve}.$$

To write numerals in base twelve it is necessary to invent new symbols in addition to using the ten symbols from the decimal system. How many new symbols are needed? Base twelve requires twelve symbols, two more than the decimal system. We can use "T" for ten and "E" for eleven as shown in the table below:

Base ten	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Base twelve	0	1	2	3	4	5	6	7	8	9	T	E	10	11	12	?	?

Notice that "T" is another way of writing 10_{ten} and "E" is another way of writing 11_{ten}. Why is 12_{ten} written as 10_{twelve}? To write 195_{twelve} in expanded notation,

$$\begin{aligned} 195_{twelve} &= (1 \times \text{twelve}^2) + (9 \times \text{twelve}^1) + (5 \times \text{one}) \\ &= (1 \times 144) + (9 \times 12) + (5 \times 1) \\ &= 257_{ten}. \end{aligned}$$

Exercises 2-8

1. Make a counting chart in base two for the numbers from zero to thirty-three.

Base Ten	1	2	3	4	. . .	33
Base Two	1	10	11		. . .	

2. Copy and complete the addition chart for base two shown at the right. How many addition facts are there?

Addition, Base Two

+	0	1
0		
1		

3. Using the same form as in Exercise 2, make a multiplication chart for base two. How many multiplication facts are there? How do the tables compare? Does this make working with the binary system difficult or easy? Explain your answer.

4. Write the following binary numerals in expanded notation and then in base ten notation.

   a. 111_{two} c. 10101_{two}

   b. 1000_{two} d. 11000_{two}

   e. 10100_{two}

5. Write in duodecimal notation the three numbers following twenty-one.

6. To write $2T0_{\text{twelve}}$ in expanded notation, we have:

   $$2T0_{\text{twelve}} = (2 \times \text{twelve}^2) + (T \times \text{twelve}) + (0 \times \text{one}).$$

   To write $2T0_{\text{twelve}}$ as a decimal numeral we would first write the latter as

   $$(2 \times 144) + (10 \times 12) + (0 \times 1).$$

   What is the decimal numeral for $2T0_{\text{twelve}}$?

7. Write the following numerals in expanded notation and then in base ten notation.

   a. 111_{twelve} c. $47E_{\text{twelve}}$

   b. $3T2_{\text{twelve}}$ d. $T0E_{\text{twelve}}$

8. Add these numbers which are expressed in binary notation. Check by expressing the numerals in the exercises, and in your answers, in decimal notation and adding the usual way.

   a. $\begin{array}{r} 101_{\text{two}} \\ \underline{10}_{\text{two}} \end{array}$ b. $\begin{array}{r} 110_{\text{two}} \\ \underline{101}_{\text{two}} \end{array}$ c. $\begin{array}{r} 10110_{\text{two}} \\ \underline{11010}_{\text{two}} \end{array}$ d. $\begin{array}{r} 10111_{\text{two}} \\ \underline{11111}_{\text{two}} \end{array}$

9. Subtract these base two numbers. Check your answers as you did in Exercise 8.

   a. $\begin{array}{r} 111_{\text{two}} \\ \underline{101}_{\text{two}} \end{array}$ b. $\begin{array}{r} 110_{\text{two}} \\ \underline{11}_{\text{two}} \end{array}$ c. $\begin{array}{r} 1011_{\text{two}} \\ \underline{100}_{\text{two}} \end{array}$ d. $\begin{array}{r} 11001_{\text{two}} \\ \underline{10110}_{\text{two}} \end{array}$

10. When people operate certain kinds of high speed computing machines, it is necessary to express numbers in the binary system. Change the following decimal numerals to base two notation:

a. 35 b. 128 c. 12 d. 100

11. Add and subtract the following duodecimal numerals. Check by expressing the numbers in decimal notation and adding and subtracting the usual way.

a. $\begin{array}{r} 2\,3\,6_{twelve} \\ \underline{T\,9_{twelve}} \end{array}$ b. $\begin{array}{r} T\,3\,2_{twelve} \\ \underline{1\,9\,3_{twelve}} \end{array}$

12. What advantages and disadvantages, if any, do the binary and duodecimal systems have as compared with the decimal system?

13. Write the following in duodecimal notation.

a. 425_{ten} b. 524_{ten}

14. BRAINBUSTER. An inspector of weights and measures carries a set of weights which he uses to check the accuracy of scales. Various weights are placed on a scale to check accuracy in weighing any amount from 1 to 16 ounces. Several checks have to be made, because a scale which accurately measures 5 ounces may, for various reasons, be inaccurate for weighings of 11 ounces and more.

What is the smallest number of weights the inspector may have in his set, and what must their weights be, to check the accuracy of scales from 1 ounce to 15 ounces? From 1 ounce to 31 ounces?

15. BRAINBUSTER. People who work with high speed computers sometimes find it easier to express numbers in the octal, or eight system rather than the binary system. Conversions from one system to the other can be done very quickly. Can you discover the method used?

Make a table of numerals as shown below:

Base ten	Base eight	Base two
1	1	1
2	2	10
5	5	101
7	?	?
15	?	?
16	?	?
32	?	?
64	?	?
256	?	?

Compare the powers of eight and two up to 256. Study the powers and the table above. $101,011,010_{two} = 532_{eight}$. Can you see why?

2-9. Summary

The decimal system has resulted from efforts of men over thousands of years to develop a workable system of notation (writing numerals). It is not a perfect system, but it has advantages other systems have not had. In this chapter you have studied some of the ancient systems, which, in their time, represented tremendous achievements in man's progress. You have also studied other systems in different bases. You have studied these systems to gain a better understanding of your own system.

In learning about other systems of notation you have learned that a number may be expressed in different numerals. For example, twelve may be written as ∩||, XII, 12_{ten}, 15_{seven}, or 1100_{two}, and so on. These numerals are not the same, yet they represent the same number. The symbols we use are not in themselves numbers. "XII" is not twelve things, nor is "10_{twelve}." They are only different numerals, or symbols for twelve.

Sometimes we confuse numbers and numerals. A number is an idea while a numeral is a symbol for the idea. We may write "2" on the blackboard to represent a set of two objects, as two students, or two books. If we erase the "2" we remove the numeral, but we do not destroy the number. In the same way, the word "pencil" is not the same as the object you hold in your hand when you are writing on paper.

The Egyptians might not have known that their system of notation was based on ten. To know this they would have had to know that it is possible to use other bases for a number system. You know this now, and you know that it is possible to use any whole number greater than one as a base. Some of these numeration systems are used. The binary system is used by electric computers. You should understand that a high speed computer is not a "brain." Rather it is a high speed slave that does only what it is told to do. High speed calculations with computers are possible because the machines operate at the speed of the flow of electricity and use large "memories" of stored information. Man was able to invent modern high speed computers because he had invented the system of writing numbers used in computer operation.

Exercises 2-9

1. Group twenty tally marks (////////////////////) to show place value for each of the number bases listed. Then write the numeral which represents twenty for each of the bases listed.
   a. **twelve** b. seven c. **five** d. **two**
2. The numerals shown below represent fifteen in various number bases. Supply the missing base for each numeral.
   a. $13_{\underline{\ ?\ }}$ b. $21_{\underline{\ ?\ }}$ c. $30_{\underline{\ ?\ }}$ d. $1111_{\underline{\ ?\ }}$
3. Write the following in expanded notation and in decimal notation:
   a. 111_{two} b. 321_{four} c. 2631_{seven} d. $37T_{twelve}$

4. Write "one thousand" as a numeral in base eight and also in base two.

5. Write the next five numerals following 88_{nine}.

*6. Babylonians used the symbols 𒁹 and 𒌋 for one and ten. By repeating these symbols they wrote fifty-nine as 𒌋𒌋𒌋 𒁹𒁹𒁹
𒌋𒌋 𒁹𒁹𒁹
𒁹𒁹𒁹

To write numbers larger than fifty-nine the Babylonians used the same symbols shown above, but they used the place value idea. Their number base was very large. It was sixty. As in our decimal system, the first place represented ones, so 𒌋𒌋𒌋 𒁹𒁹𒁹 meant (59 × 1).
𒌋𒌋 𒁹𒁹𒁹
𒁹𒁹𒁹

The second place had a value of sixty, so 𒌋𒁹𒁹 𒁹𒁹 meant (𒌋𒁹𒁹 × sixty) + (𒁹𒁹 × one)

$$= (12 \times 60) + (2 \times 1)$$
$$= 720 + 2$$
$$= 722$$

The first three place values in the Babylonian system were:

	sixty^2	sixty	one
Meaning in base ten	60 × 60	60	1

a. The early Babylonians did not have a symbol for zero. They sometimes left empty spaces where we write zeros, as 𒁹 𒁹 . This meant the reader had to guess from the content of the reading whether the numeral 𒁹 𒁹 meant (𒁹 + 𒁹) × one
or (𒁹 × sixty) + (𒁹 × one).
or (𒁹 × sixty^2) + (𒁹 × one).
Write three decimal numerals for the Babylonian numeral
𒁹 𒁹

b. Does the decimal numeral "11" have more than one possible meaning? Why?

c. Find the decimal value for 𒁹𒁹 𒌋𒁹𒁹 𒌋𒁹

= (𒁹𒁹 × _?_) + (𒌋𒁹𒁹 × _?_) + (𒌋𒁹 × _?_)

= ? + ? + ?

= ?

d. Write the decimal numerals 70, 111, and 4000 in Babylonian notation.

e. Write 𒌋𒁹 𒌋𒁹 + 𒁹𒁹𒁹 𒌋𒌋 as a Babylonian numeral.

*7. In Lincoln's Gettysburg Address, the term "four score and seven" is used. What number base did he use? Write the decimal numeral for this number.

*8. A positional number system uses the symbols 0, A, B, C, and D to represent the numbers from zero to four. If these are the only symbols used in the system, write the decimal numeral for D C B A 0.

*9. A base ten number consists of three digits, 9, 5, and another in that order. If these digits are reversed and then subtracted from the original number, an answer will be obtained consisting of the same digits arranged in a different order still. What is that digit?

10. BRAINBUSTER. Suppose a place value number system uses the capital letters of the alphabet (A, B, C, . . . Y, Z.) as symbols for numerals. The letter "O" is removed from its regular position between N and P and is used as the symbol for zero. What is the base for this system of notation? What decimal numeral is represented by "B E" in this base? by "T W O"? by "F O U R"?

11. BRAINBUSTER. There are a number of ways to change numerals written in other number bases to base ten notation. A student suggested this method:

Example A: To change 46_{twelve} to base ten notation. Because there are 2 more symbols in base twelve, multiply (2 × 4) and add the result to 46_{ten}.

Does this method work for 46_{twelve}? Does it work for any two digit number written in base twelve?

Example B: To change 46_{seven} to base ten notation. Because there are three fewer symbols, multiply (3 × 4) and subtract from 46_{ten}. Does the method work for 46_{seven}? Does it work for any two digit number written in base seven?

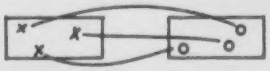
Chapter 3

WHOLE NUMBERS

3-1. Counting Numbers

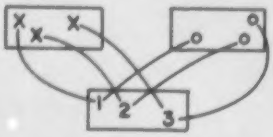
The counting numbers are the numbers used to answer the question "How many?" Primitive man developed the idea of number by the practice of matching objects, or things, in one set with objects in another set. When a man's sheep left the fold in the morning he could put a stone in a pile as each sheep went out. When the sheep returned in the evening he took a stone out of the pile as a sheep went into the pen. If there were no stones left in the pile when the last sheep was in the pen he knew that all the sheep had returned. Similarly, in order to keep count of the number of wild animals he had killed he could make notches in a stick -- one notch for each animal. If he were asked how many animals he had killed he could point to the notches in the stick. The man was saying that there were just as many animals killed as there were notches in the stick. The man was trying to answer the question "How many?" by making a one-to-one correspondence between the animals and the notches in the stick. He was also trying to answer the question "How many?" by making a one-to-one correspondence between the stones of the pile and the sheep of the flock. The one-to-one correspondence means that exactly one stone corresponded to each sheep and exactly one sheep corresponded to each stone. This says that the number of sheep was the same as the number of stones.

Some of us have learned the meaning of number in counting by using such one-to-one correspondences. We look at various sets of objects as in the figure. We see that there is a certain property that these sets possess. This property may be described by saying that there are "just as many" marks in one set as in the other. A one-to-one correspondence between the sets can be shown by joining the



marks with strings, or lines. Each mark is joined to a mark of the other set. No marks are left over in either set and no mark is used twice. The correspondence shows that there are "just as many" marks in one set as in the other but it does not tell us "how many" there are in terms of a number.

Fortunately we have a standard set which we can use to tell us "how many" there are in each set. It also can be used to tell us that there are "just as many" in one set as in the other. This standard set is the set of counting numbers represented by the numerals 1, 2, 3, 4, 5, . . . In the figure each set of marks is put in a one-to-one correspondence with the set of numerals 1, 2, 3. The number of marks is the same as the number represented by the last numeral of the matching set. This kind of one-to-one correspondence between the marks and the set of numerals tells us that there are "just as many" in one set as in the other, and also tells us "how many" marks are in each set.



The method of using the counting numbers is such a natural one that the counting numbers are also called the "natural numbers." In this text we call them counting numbers. You may see them called "natural numbers" in other books.

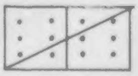
Let us agree that our first counting number is 1. If we wish to talk about all the counting numbers and zero we call this set of numbers the "whole numbers."

Exercises 3-1

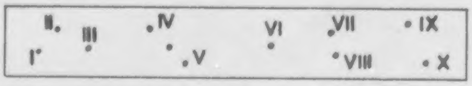
1. Rearrange these numerals so that the numbers they represent are in order of size. Place the smallest first.
   a. 1, 2, 3, 6, 4, 5
   b. 2 + 1, 1 + 1, 3 + 1, 0 + 5, 1 + 6, 5 + 1, 5 + 3
   c. IV, XI, V, VI, VII, VIII, X, IX
   d. $(3)_{\text{seven}}$, $(10)_{\text{seven}}$, $(4)_{\text{seven}}$, $(1)_{\text{seven}}$, $(5)_{\text{seven}}$, $(2)_{\text{seven}}$, $(6)_{\text{seven}}$

[sec. 3-1]

2. Which of the numbers represented by the numerals 2, 5, 7, 8 are counting numbers between one and ten? Between six and eleven?
3. Is it possible to arrange 24 marks so that one can tell how many there are without counting each one? Explain or show.
4. Rearrange the following marks so that the number of marks may be more easily determined.
   a. | | | | | | | | | |
   b. - - - - - - - - - -
5. Explain two ways of finding the number of dots in the figure without counting each dot.



6. Try to find out the number name for the first counting number in some other languages. (French, Spanish, German, Russian, etc.)
7. In counting the number of dots in the figure a mistake was made. What was it?



8. Was there a mistake made in counting the number of x's in this figure? If yes, what was it?

   ⊗ ⊗ ⊗ ⊗ ⊗ ⊗ ⊗ ⊗ ⊗ ⊗ ⊗ ⊗ ⊗
   ① ② ③ ④ ⑤ ⑥ ⑦ ⑨ ⑩ ⑪ ⑫ ⑬ ⑭

9. Suppose 16 tickets were sold and the first one had the numeral 2 on it. What was the numeral on the last ticket if they were sold in order?
10. A theatre owner wants to know how many people attended his theatre last night. He knows the first ticket was marked 27 and the last ticket was marked 81. How did he figure that 54 people attended? Was he correct?
11. If there is a one-to-one correspondence between the set of people in the room and the set of pairs of shoes in the room, then there is a two-to-one correspondence between the set of

shoes in the room and the set of people in the room. List a few examples of two-to-one and four-to-one correspondences.

12. The following illustrates a one-to-one correspondence between the __________ numbers and the __________ numbers.

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & . & . & . \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & & \\ 2 & 4 & 6 & 8 & 10 & 12 & . & . & . \end{array}$$

3-2. Commutative Properties for Whole Numbers

If you have three apples in a basket and put in two more, then the number of apples in the basket is obtained by adding 2 to 3. You think of 3 + 2. If you started with two apples in the basket, and put in three more, then the number of apples in the basket is obtained by adding 3 to 2. You think of 2 + 3. In either case it is clear that there will be 5 apples in the basket. We may write 2 + 3 = 3 + 2.

The arithmetic teacher read two large numbers to be added. One boy did not understand what his teacher said when she read the first number. He wrote the second number and then asked her to repeat the first number. When she read it again, he wrote it below the second number instead of above it. If all the students do the addition correctly, will the boy find the same sum as the students who heard all the dictation the first time?

The boy wrote:	2437 6254	The others wrote:	6254 2437

We call this idea which was just described the commutative property of addition for whole numbers. It means that the order in which we add two numbers does not affect the sum. The word property is used here in the usual meaning of the word -- it is something that belongs to the operation of addition:

3 added to 4 is 7 or 4 + 3 = 7,
4 added to 3 is 7 or 3 + 4 = 7.

Thus, we can write 4 + 3 = 3 + 4. This checks the commutative property of addition for these two whole numbers.

The commutative property of addition for whole numbers may be stated as:

Property 1. If a and b represent whole numbers then

$$a + b = b + a.$$

In the above example a is 4 and b is 3.

Multiplication is another operation which we perform on numbers. Is there a commutative property of multiplication? Let us see how to find the answer to the question.

Suppose we have five rows of chairs with 3 chairs in each row. Then, suppose we decide to change the arrangement to make three rows with 5 chairs in each row. Will we need more chairs? Will we have any chairs which are not used in the second arrangement?

```
***                              *****
***                              *****
***                              *****
***
***                              3 rows of 5 each:   3 × 5 = 15

5 rows of 3 each:   5 × 3 = 15
```

In learning the multiplication tables you learned that 7 × 5 = 35 and that 5 × 7 = 35. Similarly 9 × 8 = 72 and 8 × 9 = 72. When the two numbers are the same, the products are the same, regardless of which number is written first.

These examples indicate that there is a commutative property of multiplication. This commutative property of multiplication for whole numbers states that the product of two whole numbers is the same whether the first be multiplied by the second or the second be multiplied by the first. We state this as:

Property 2. If a and b represent whole numbers, then

$$a \times b = b \times a.$$

We can use this property to detect mistakes which we might make in multiplying one number by another. We found these products:

$$\begin{array}{r} 436 \\ \underline{125} \\ 2180 \\ 872 \\ \underline{436} \\ 54500 \end{array} \qquad\qquad \begin{array}{r} 125 \\ \underline{436} \\ 730 \\ 365 \\ \underline{600} \\ 63380 \end{array}$$

In this computation the commutative property shows that we have made at least one mistake. Find all the mistakes.

In both Property 1 and Property 2 we used letters to represent numbers. This idea of using letters to stand for any number whatsoever in stating general principles is a very useful part of mathematical language. Sometimes the letter x and the multiplication sign may be mistaken for each other, so we often use a raised dot, $\cdot$, to indicate multiplication. For example we can write $4 \cdot 3$ for 4×3 and $a \cdot b$ for $a \times b$.

Many symbols are used to simplify the writing of mathematics. Any symbol can be introduced and used if we first decide what the symbol is to mean and always use it to have that meaning. The use of the raised dot is a good example.

In mathematics we often say that one number is greater than another. To simplify writing the phrase "is greater than" we use the symbol $>$. So, to write "5 is greater than 3" we merely write $5 > 3$. To indicate that "a is greater than b" we write $a > b$. Similarly, we use the symbol $<$ to mean "is less than." Hence, we write $4 < 7$ for "4 is less than 7." Notice that each of these new symbols points toward the smaller of the two numbers being compared.

Sometimes we merely wish to note that two numbers are not equal. The symbol $\neq$ is used for "is not equal to." For example, $5 \neq 3$ and $4 \neq 0$.

In comparing three numbers such as 3, 6 and 11, we may write $3 < 6 < 11$ or $11 > 6 > 3$. Note that the statement $3 < 6 < 11$ really stands for the two statements "3 is less than 6" and "6 is less than 11."

Exercises 3-2a

1. Indicate whether each statement is true or false:
   - a. $6 + 4 = 4 + 6$
   - b. $13_{\text{four}} + 32_{\text{four}} < 32_{\text{four}} + 13_{\text{four}}$
   - c. $6 < 7 < 14$
   - d. $1 + 5 = 5 + 1$
   - e. $6 \cdot 5 = 5 \cdot 7$
   - f. $6 + 3 = 4 + 5$
   - g. $45 \cdot 36 < 36 \cdot 45$
   - h. $5 + 4 > 5 + 3$
   - i. $315 + 462 = 462 + 315$
   - j. $5 > 3 > 10$
   - k. $8 \div 2 = 2 \div 8$
   - l. $851 + 367 = 158 + 763$
   - m. If $16 > 7$ and $7 > 5$ then $16 > 5$

2. Add. Then use the commutative property to check addition.
   - a. $\begin{array}{r}465\\ \underline{179}\end{array}$
   - b. $\begin{array}{r}37461\\ \underline{73135}\end{array}$
   - c. $\begin{array}{r}73967\\ \underline{81785}\end{array}$
   - d. $\begin{array}{r}43_{\text{seven}}\\ \underline{32_{\text{seven}}}\end{array}$

3. Using the symbols $=$, $<$, and $>$, make the following true.
   - a. $7 + 4 \ ? \ 4 + 7$
   - b. $12 \cdot 5 \ ? \ 5 \cdot 11$
   - c. $23 \cdot 12 \ ? \ 12 \cdot 32$
   - d. $3 \ ? \ 6$
   - e. $16 \ ? \ 9 \ ? \ 3$
   - f. $(3 \cdot 2) + 5 \ ? \ 5 + (3 \cdot 2)$
   - g. $8 - 3 \ ? \ 9 - 3$
   - h. $86 \cdot 135 \ ? \ 135 \cdot 86$
   - i. $24 \div 3 \ ? \ 3 \div 24$
   - j. Given that a, b, and c are whole numbers: If $a > b$ and $b > c$, then $a \ ? \ c$.

4. Multiply. Then use the commutative property to check the multiplication.
   - a. $\begin{array}{r}36\\ \underline{57}\end{array}$
   - b. $\begin{array}{r}305\\ \underline{84}\end{array}$
   - c. $\begin{array}{r}476\\ \underline{609}\end{array}$
   - d. $\begin{array}{r}31_{\text{seven}}\\ \underline{25_{\text{seven}}}\end{array}$

5. Give the whole number or whole numbers which may be used in place of a to make the statements true.
   - a. $3 + a = 3 + 5$
   - b. $5 \cdot 7 = 7 \cdot a$
   - c. $2 \cdot a < 2 \cdot 1$
   - d. $3 \cdot a < 3 \cdot 2$
   - e. $132 + a = 46 + 132$
   - f. $2 + a < 2 + 7$
   - g. $7 \cdot 3 > a \cdot 5$
   - h. $a + 3 = 3 + a$

The commutative properties of addition and multiplication have been stated in symbolic form:

$$a + b = b + a \quad \text{and} \quad a \cdot b = b \cdot a.$$

Notice how similar the statements are.

Do you think subtraction has the commutative property? To find out we must ask whether $a - b$ is equal to $b - a$ for all whole numbers a and b. If we can find at least one pair of whole numbers for which it is not true, then subtraction cannot have the commutative property. Is $6 - 9$ equal to $9 - 6$? No. In fact, $(9 - 6)$ is 3 and there is no whole number which is $(6 - 9)$.

Exercises 3-2b

1. Interchange the numbers in each of the following. In which ones is the result unchanged?
   - a. $1 + 2$
   - b. $6 + 8$
   - c. $7 \cdot 9$
   - d. $4 - 5$
   - e. $12 \div 3$
   - f. $9 \div 4$
   - g. $5 - 4$
   - h. $3 \div 12$
   - i. $4 \div 9$
2. Does division of whole numbers have the commutative property? Give an example which illustrates your answer.
3. Which of the following activities are commutative?
   - a. To put on a hat and then a coat.
   - b. To put on socks and then shoes.
   - c. To pour red paint into blue paint.
   - d. To close the hatch and dive the submarine.
   - e. To put on your left shoe and then the right shoe.
4. We shall invent the operation "M" which shall mean to choose the larger of two numbers. If the numbers are the same we shall choose that one number. Is the operation commutative? Example: $3 \text{ M } 4 = 4$
5. Which of the defined operations below are commutative?
   - a. "D" means to find the sum of the first and twice the second. Example: $3 \text{ D } 5 = 3 + (2 \cdot 5)$ or 13.
   - b. "Z" means to find the sum of the first and the product of the first and the second. Example: $4 \text{ Z } 7 = 4 + (4 \cdot 7)$ or 32.

c. "F" means to find the product of the first and one more than the second. Example: 8 F 0 = 8 · 1 or 8.

d. "Q" means to find three times the sum of the first and the second. Example: 8 Q 5 = 3 · (8 + 5) or 39.

6. List some activities which are commutative and some which are not commutative.

3-3. Associative Properties for Whole Numbers

What is meant by 1 + 2 + 3? Do we mean (1 + 2) + 3 in which we add 1 and 2 and then add 3 to the sum? Or do we mean 1 + (2 + 3) in which we add 2 and 3 and then add their sum to 1? Or, does it make any difference? We have seen that the order in which two numbers are added does not affect the sum (commutative property of addition). Now we see that the way we group three numbers to add them does not affect the sum. For example,

$$(1 + 2) + 3 = 3 + 3 = 6 \quad \text{and}$$
$$1 + (2 + 3) = 1 + 5 = 6.$$

We call this idea of grouping the numbers differently without changing the sum the associative property of addition for whole numbers. This property may be used to make addition easier if the sum of one pair of three numbers is easier to find than the sum of another pair. If you are asked to add 12 + 4 + 2 you might first add 12 and 4 and then add 2 to 16. Or you might think of first adding 4 and 2 and then adding 6 to 12. If we add each of the following by grouping the numbers differently we will be showing applications of the associative property.

$$7 + 9 + 11 = 7 + (9 + 11) = 7 + 20 = 27$$
$$12 + 7 + 33 = 12 + (7 + 33) = 12 + 40 = 52$$
$$97 + 53 + 100 = (97 + 53) + 100 = 150 + 100 = 250$$

The associative property can be used in finding the sum of 12 and 7. Perhaps you have always used it but did not call it by name. Notice how it can be used: $12 + 7 = (10 + 2) + 7 = 10 + (2 + 7) = 19$.

Just as we stated the commutative property of addition, we now state the associative property of addition.

Property 3. If a, b and c represent any whole numbers

$$(a + b) + c = a + (b + c).$$

In everyday life we speak of "adding" or combining several things. Whether such combinations have the associative property depends on the things we combine. Is (gasoline + fire) + water the same as gasoline + (fire + water)

The commutative property of addition means we may change the order of any two numbers without affecting the sum. The associative property means that we may group numbers in pairs for the purpose of adding pairs of them without affecting the sum. Just as there is a commutative property for addition and multiplication, we might expect the associative property to belong to both operations.

What is meant by $2 \cdot 5 \cdot 4$? Do we mean $(2 \cdot 5) \cdot 4$ in which we first multiply 2 by 5 and then multiply 10 by 4, or do we mean $2 \cdot (5 \cdot 4)$ in which we first multiply 5 by 4 and then multiply 2 by 20? Both give the same answer and we conclude that we can give either meaning to $2 \cdot 5 \cdot 4$. This is true for any whole numbers.

Property 4. If a, b, and c represent any whole numbers,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

This is the symbolic statement of the associative property of multiplication for whole numbers.

Sometimes it is convenient to rearrange the order of the numbers which are to be added, or multiplied, in order to make the operation easier. This may be done by use of the commutative property. Then the addition or multiplication can be performed by grouping the numbers according to the associative property. The following examples are illustrations of the uses of both properties in the same problem.

$$17 + (19 + 13) = 17 + (13 + 19) = (17 + 13) + 19 = 30 + 19 = 49$$

$$50 \cdot (17 \cdot 4) = 50 \cdot (4 \cdot 17) = (50 \cdot 4) \cdot 17 = 200 \cdot 17 = 3400$$

Is there an associative property for subtraction? Perhaps we can answer the question by considering just one example. We try 10 - (6 - 4) which is 10 - 2 or 8. But (10 - 6) - 4 = 0, so that 10 - (6 - 4) is not equal to (10 - 6) - 4. This shows that subtraction does not have the associative property. At first you may think that one example is not enough and that the property might hold if we used some other numbers. But, if the associative property is to hold for subtraction then it must hold for all whole numbers. Hence, by showing one set of three whole numbers for which the property is not true we know that it cannot be a property for all whole numbers.

Do you think the associative property holds for division? What does 16 ÷ 4 ÷ 2 mean? We cannot tell. It may mean (16 ÷ 4) ÷ 2, or it may mean 16 ÷ (4 ÷ 2). The first of these equals 2 and the second equals 8, so they are not equal to each other. This shows that division does not have the associative property.

These remarks about subtraction and division show us also that expressions like 10 - 6 - 4 and 16 ÷ 4 ÷ 2 do not have any meaning. Of course, the expressions, (10 - 6) - 4 and 10 - (6 - 4), do have meanings and they are different. Also, (16 ÷ 4) ÷ 2 and 16 ÷ (4 ÷ 2) make sense, but their meanings are different.

Exercises 3-3

1. Example: (4 + 3) + 2 = 4 + (3 + 2)
   Here, (4 + 3) + 2 = 7 + 2 = 9, and 4 + (3 + 2) = 4 + 5 = 9. This illustrates the associative property of addition. Show that the following are true in the above way. State the property illustrated in each problem.
   a. (4 + 7) + 2 = 4 + (7 + 2)
   b. 8 + (6 + 3) = (8 + 6) + 3
   c. 46 + (73 + 98) = (46 + 73) + 98

d. $(6 \cdot 5) \cdot 9 = 6 \cdot (5 \cdot 9)$

e. $(21 + 5) + 4 = 21 + (5 + 4)$

f. $(9 \cdot 7) \cdot 8 = 9 \cdot (7 \cdot 8)$

g. $436 + (476 + 1) = (436 + 476) + 1$

h. $(57 \cdot 80) \cdot 75 = 57 \cdot (80 \cdot 75)$

2. a. Does $(10 - 7) - 2$ equal $10 - (7 - 2)$?

b. Does $18 - (5 - 2)$ equal $(18 - 5) - 2$?

c. What generalization can you make regarding the associative property of subtraction?

3. a. Does $(32 \div 8) \div 2$ equal $32 \div (8 \div 2)$?

b. Does $(60 \div 30) \div 2$ equal $60 \div (30 \div 2)$?

c. Place parentheses in $75 \div 15 \div 5$ so that it will equal 1.

d. Place parentheses in $75 \div 15 \div 5$ so that it will equal 25.

e. Place parentheses in $80 \div 20 \div 2$ so that it will equal 8.

f. Place parentheses in $80 \div 20 \div 2$ so that it will equal 2.

g. What generalization can be made concerning the associative property and division?

4. Rewrite these problems using the associative and commutative properties whenever they make the operation easier. Use parentheses to show the operation which is performed first. Find the answers. Example:

$$\begin{aligned} & 25 + (36 + 75) \\ = {} & 25 + (75 + 36) && \text{by commutative property of addition} \\ = {} & (25 + 75) + 36 && \text{by associative property of addition} \\ = {} & 100 + 36 \\ = {} & 136 \end{aligned}$$

a. $(6 + 1) + 9$

b. $2 \cdot (13 \cdot 10)$

c. $(12 \cdot 9) \cdot 10$

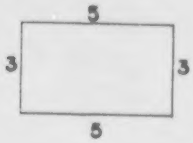
d. $4 \cdot (25 \cdot 76)$

e. $340 + (522 + 60)$

f. $(5 \cdot 67) \cdot 2$

3-4. The Distributive Property

In finding the perimeter of the top of a desk one pupil measured the length of each side in feet and found the measurements as shown in the diagram. Then he found the perimeter in feet by finding the sum 5 + 3 + 5 + 3 = 16. Another pupil said he thought that this was all right but that it was more work than necessary. He said he would add 5 and 3 and multiply their sum by 2. Will this give the same answer? A third pupil said she thought it would be better to multiply 5 by 2 and 3 by 2 and then add these two products. The second and third pupils may not have known the name of the principle they were using but it is useful and important. It is called the distributive property. In terms of the pupils' problem it states simply that



$$2 \cdot (5 + 3) = (2 \cdot 5) + (2 \cdot 3)$$

and

$$2 \cdot (5 + 3) = (2 \cdot 8)$$

Eight girls and four boys are planning a skating party. Then, each girl invites another girl and each boy invites another boy. The original number of girls has been doubled. The original number of boys has been doubled. Has the total number of children been doubled or not? Let us see. In all there will be (2 · 8) girls and (2 · 4) boys or a total of (2 · 8) + (2 · 4) = 24 children at the party. Let us look at this another way. When the party was planned, there were (8 + 4) = 12 children. The final number of children is 2 · (8 + 4) or 2 · 12.

We have seen that $(2 \cdot 8) + (2 \cdot 4) = 16 + 8 = 24$

and $2 \cdot (8 + 4) = 2 \cdot 12 = 24.$

So we can write $(2 \cdot 8) + (2 \cdot 4) = 2 \cdot (8 + 4)$.

You have been using this property in many ways for a long time. Consider, for example, 3 · 13 or $\begin{array}{r}13.\\ \times\ 3\\ \hline 39\end{array}$ You were really using the distributive property because:

$$3 \cdot 13 = 3 \cdot (10 + 3) = (3 \cdot 10) + (3 \cdot 3) = 30 + 9 = 39.$$

Let us see how you use the distributive property in finding the product $9 \cdot 36$. You probably perform the multiplication about as follows:

$$\begin{array}{r} 36 \\ \times\ 9 \\ \hline 324 \end{array} \qquad \text{or} \qquad \begin{array}{rl} 36 & \\ \times\ 9 & \\ \hline 54 & (9 \times 6) \\ 270 & (9 \times 30) \\ \hline 324 & \end{array}$$

Do you see that the left example is a short way of doing the problem? You were really using the distributive property:

$$\begin{aligned} 9 \cdot 36 &= 9 \cdot (30 + 6) \\ &= (9 \cdot 30) + (9 \cdot 6) \quad \text{distributive property} \\ &= 270 + 54 \\ &= 324. \end{aligned}$$

The distributive property is also important in operations involving fractions. Let us find the product of 8 and $12\frac{1}{4}$. First, recall that $12\frac{1}{4}$ means $12 + \frac{1}{4}$. Then

$$\begin{aligned} 8 \cdot 12\tfrac{1}{4} &= 8 \cdot (12 + \tfrac{1}{4}) \\ &= (8 \cdot 12) + (8 \cdot \tfrac{1}{4}) = 96 + 2 \\ &= 98. \end{aligned}$$

The distributive property is:

Property 5. If a, b, and c are any whole numbers then

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c).$$

The distributive property is the only property of the three we have studied in this chapter which involves two operations, namely, addition and multiplication. This does not mean that any problem which involves these two operations is performed by using the distributive property. For example, $(3 \cdot 5) + 14$ means that the product of 3 and 5 must be found and then 14 added to the product: $(3 \cdot 5) + 14 = 15 + 14 = 29$.
However, $3 \cdot (5 + 14) = (3 \cdot 5) + (3 \cdot 14) = 15 + 42 = 57$.

The commutative property of multiplication permits us to write $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$. Let us see why. First

$(b + c) \cdot a = a \cdot (b + c)$, commutative property

and $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$, distributive property.

Therefore, $(b + c) \cdot a = (a \cdot b) + (a \cdot c)$

Also, $(a \cdot b) = (b \cdot a)$ commutative property

$(a \cdot c) = (c \cdot a)$ commutative property.

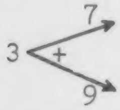
Hence, $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$.

This justifies the multiplication of $12\frac{1}{4}$ by 8 in the form

$$12\tfrac{1}{4} \cdot 8 = (12 + \tfrac{1}{4}) \cdot 8 = (12 \cdot 8) + (\tfrac{1}{4} \cdot 8)$$
$$= 96 + 2 = 98.$$

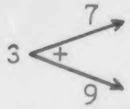
Some of you may wish to use a sketch to help you remember that the first factor is distributed over all the numbers being added in the second factor. One such sketch is illustrated here. For example, consider the product $3 \cdot (7 + 9)$.

Sketch:



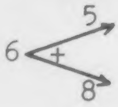
$(3 \cdot 7)$

$(3 \cdot 9)$ $= 21 + 27 = 48.$

Note that  denotes that the 3 multiplies both the 7 and the 9 and that these products are then added. The arrows always point to the numbers that are to be added. Another example might be:

$$6 \cdot (5 + 8)$$

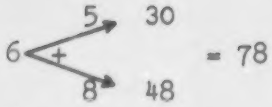
Sketch:



$(6 \cdot 5)$

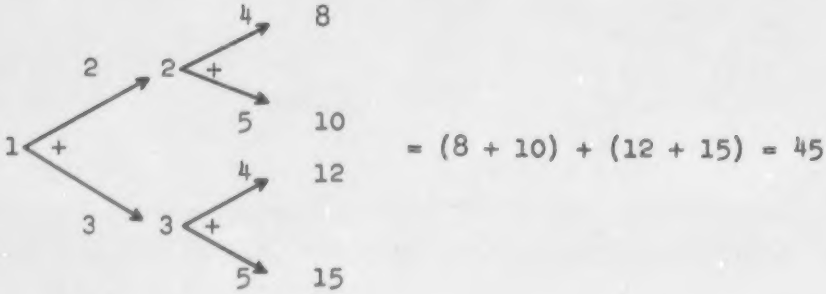
$(6 \cdot 8)$ $= 30 + 48 = 78$

or



Another example. (This may help you with problem 7 in the exercises.)

$$(2 + 3) \cdot (4 + 5)$$



Exercises 3-4

1. Use the sketch method illustrated above to do the indicated operations.

   a. $5 \cdot (6 + 4)$
   b. $3 \cdot (9 + 6)$
   c. $12 \cdot (6 + 7)$
   d. $9 \cdot (13 + 17)$
   e. $(6 + 4) \cdot (8 + 7)$
   f. $(20 + 7) \cdot (10 + 4)$

2. Show that the following are true by doing the indicated operations. Example: $3 \cdot (4 + 3) = (3 \cdot 4) + (3 \cdot 3)$

$$3 \cdot (4 + 3) = 3 \cdot 7 = 21$$
$$(3 \cdot 4) + (3 \cdot 3) = 12 + 9 = 21$$

   a. $4 \cdot (7 + 5) = (4 \cdot 7) + (4 \cdot 5)$
   b. $(3 \cdot 6) + (4 \cdot 6) = 6 \cdot (3 + 4)$
   c. $(8 \cdot 6) + (7 \cdot 6) = (8 + 7) \cdot 6$
   d. $23 \cdot (2 + 3) = (23 \cdot 2) + (23 \cdot 3)$
   e. $11 \cdot (3 + 4) = (11 \cdot 3) + (11 \cdot 4)$

f. $(6 \cdot 5) + (6 \cdot 3) = 6 \cdot (5 + 3)$

g. $2 \cdot (16 + 8) = (2 \cdot 16) + (2 \cdot 8)$

h. $12 \cdot (5 + \frac{1}{4}) = (12 \cdot 5) + (12 \cdot \frac{1}{4})$

i. $(67 \cdot 48) + (67 \cdot 52) = 67 \cdot (48 + 52)$

j. $(72 \cdot \frac{1}{2}) + (\frac{1}{2} \cdot 72) = 72 \cdot (\frac{1}{2} + \frac{1}{2})$.

3. Make each of the following a true statement illustrating the distributive property.

a. $3 \cdot (4 + \quad) = (3 \cdot 4) + (3 \cdot 3)$

b. $2 \cdot (\quad + 5) = (2 \cdot 4) + (\quad \cdot 5)$

c. $13 \cdot (6 + 4) = 13 \cdot (\ ) + 13 \cdot (\ )$

d. $(2 \cdot 7) + (3 \cdot \ ) = (\qquad) \cdot 7$

e. $(\quad \cdot 4) + (\quad \cdot 4) = (6 + 7) \cdot (\ )$.

4. Using the distributive property rewrite each of the following.

Examples: 1. $5 \cdot (2 + 3) = (5 \cdot 2) + (5 \cdot 3)$

2. $(6 \cdot 4) + (6 \cdot 3) = 6 \cdot (4 + 3)$

a. $(9 \cdot 8) + (9 \cdot 2)$

b. $8 \cdot (14 + 17)$

c. $12 \cdot (5 + 7)$

d. $(13 + 27) \cdot 6$

e. $15 \cdot (6 + 13)$

f. $(5 \cdot 12) + (4 \cdot 12)$

5. Using the idea of the distributive property we can rewrite, for example:

(1) $10 + 15$ as $(5 \cdot 2) + (5 \cdot 3)$ or $5 \cdot (2 + 3)$

(2) $15 + 21$ as $(3 \cdot 5) + (3 \cdot 7)$ or $3 \cdot (5 + 7)$

Use the distributive property to rewrite the following in a similar way.

a. $35 + 40$

b. $12 + 15$

c. $55 + 10$

d. $27 + 51$

e. $100 + 115$

f. $30 + 21$

6. Which of the following are true?

a. $3 + (4 \cdot 2) = (3 + 4) \cdot (3 + 2)$

b. $3 \cdot (4 - 2) = (3 \cdot 4) - (3 \cdot 2)$

c. $(4 + 6) \cdot 2 = (4 \cdot 2) + (6 \cdot 2)$

d. $(4 + 6) \div 2 = (4 \div 2) + (6 \div 2)$

e. $3 + (4 \quad 2) = (3 \cdot 4) + (3 \cdot 2)$

*7. We can write 45 as $(40 + 5)$ and 23 as $(20 + 3)$. Using the distributive property the product of 45 and 23 would be:

$(40 + 5) \cdot (20 + 3)$ or

$40 \cdot (20 + 3) + 5 \cdot (20 + 3)$ or

$(40 \cdot 20) + (40 \cdot 3) + (5 \cdot 20) + (5 \cdot 3)$

Check

$$\begin{array}{r} 45 \\ \times\ 23 \\ \hline 135 \\ 90 \\ \hline 1035 \end{array}$$

Completing the operations gives:

$800 + 120 + 100 + 15$ or 1035

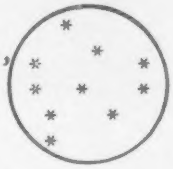
Rewrite the following using the distributive property and check as above. Try the sketch method with (a) and (d).

a. $27 \cdot 34$
b. $13 \cdot 22$
c. $37 \cdot 33$
d. $64 \cdot 66$
e. $75 \cdot 75$
f. $21 \cdot 29$

8. BRAINBUSTER. Indicate which property was used in going from one line to the next.

a. $[(2 \cdot 5) + (3 \cdot 2)] + (5 \cdot 2) \cdot 7$
b. $[(5 \cdot 2) + (3 \cdot 2)] + (5 \cdot 2) \cdot 7$ commutative property for multiplication
c. $[(5 \cdot 2) + (3 \cdot 2)] + 5 \cdot (2 \cdot 7)$ "?"
d. $[(3 \cdot 2) + (5 \cdot 2)] + 5 \cdot (2 \cdot 7)$ "?"
e. $(3 \cdot 2) + [(5 \cdot 2) + 5 \cdot (2 \cdot 7)]$ "?"
f. $(3 \cdot 2) + [(5 \cdot 2) + 5 \cdot (7 \cdot 2)]$ "?"
g. $(3 \cdot 2) + [(5 \cdot 2) + (5 \cdot 7) \cdot 2]$ "?"
h. $(3 + [5 + (5 \cdot 7)]) \cdot 2$ "?"

3-5. Sets and the Closure Property



If you wished to refer to the stars in the figure, how would you do it? Would you say the set of stars? The collection of stars? The group of stars? If you speak of the pupils in a certain classroom you may say, "the class in Miss Johnson's room." Would "the set of pupils in Miss Johnson's room" convey the same meaning? How about "the collection of pupils in Miss Johnson's room"? Either of the words, *collection*, or *set* may be used here. They have the same meaning. (If you are studying French, it may occur to you that *ensemble* could be used.) When we need such a word in mathematics, we will use the word *set*, as a set of numbers, a set of marks on the page, a set of stars in a diagram.

A set of numbers: 5, 36, 7, 8

A set of marks: ////////

A set of stars in a diagram:

Other examples of sets are: the set of coins in your pocket, the set of vowels in our alphabet, a set of chessmen, a set of cattle (you might say a herd of cattle), the set of cities in the U.S.A. which have a population of more than one million.

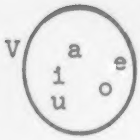
The counting numbers form a set. Remember that the counting numbers are 1, 2, 3, 4, 5, 6, ... where the three dots are used to indicate that the set of numbers continues indefinitely. There is no last number. We are going to use N to represent the set of counting numbers and we will put the counting numbers within braces { } to indicate that they are the objects in the set which is designated by N. Hence, we may write

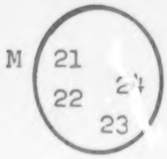
$$N = \{1, 2, 3, \ldots\}$$

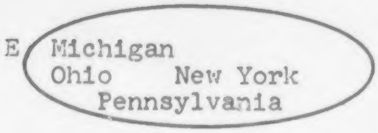
and read it "N is the set of counting numbers."

We may choose any capital letter to represent the set. If we have the set $S = \{1, 2, 3, 4, 5, 6, 7\}$ we may describe this by saying that S is the set of counting numbers from 1 to 7 inclusive, or S is the set consisting of the counting numbers less than 8.

A few more examples of sets and the abbreviated way of writing them will help make the concept clear. "V is the set of vowels in our alphabet" becomes "$V = \{a, e, i, o, u\}$." "M is the set of counting numbers which are greater than 20 and less than 25" becomes "$M = \{21, 22, 23, 24\}$." "E is the set of states in the U. S. A. which are touched by Lake Erie" becomes "E = {Michigan, Ohio, Pennsylvania, New York}." A way of illustrating each one of these may be helpful.







In the first set the objects are letters, in the second the elements are numbers, in the third each element is a state. We use the word *element* for any object of a set. Thus, an element may be a letter, a number, a word, a cat, a marble or whatever is in the set we are thinking about.

We are now going to use the set of counting numbers to help us understand another new idea for sets. This is the idea of *closure*. If we add any two counting numbers, the sum is a certain counting number. For example, 7 + 9 = 16, 234 + 543 = 777 and each sum is a counting number. If the sum of any two elements of a set is an element of the set, we say that the set is *closed* under addition. Since the sum of any two counting numbers is a counting number, the set N of counting numbers is closed under addition. It must be emphasized here that *any two* means *every two*. The set S = {1, 2, 3, 4, 5, 6, 7} is not closed under addition since we can find two numbers in the set whose sum is not in the set; e.g., 5 + 6 = 11 and 11 is not in S. Is the set M = {21, 22, 23, 24} closed under addition? Give the reason for your answer. Notice that if there is at least one pair of elements in M whose sum is not in M, then M is not closed under addition.

Closure deals with a property of sets under a given operation. The set need not be the counting numbers. The operation may not be addition. For example, let T be the set of all counting numbers ending in 0 or 5. This set is closed under multiplication. It is not closed under division since, for example, (20 ÷ 5) is not an element of T.

Exercises 3-5-a

1. Let Q = {1, 3, 5, 7, 9, 11, 13, . . .} be the set of all odd numbers.
   - a. Is the sum of two odd numbers always an odd number?
   - b. Is the set of odd numbers closed under addition?
2. Is the set of even numbers closed under addition?
3. Is the set of all multiples of 5 (5, 10, 15, 20, 25, etc.) closed under addition?
4. What is true of the sets of numbers in Exercises 1, 2, 3 under multiplication?
5. Are the following sets of numbers closed under addition?
   - a. The set of counting numbers greater than 50?
   - b. The set of counting numbers from 100 through 999?
   - c. The set of counting numbers less than 48?
   - d. The set of counting numbers whose numerals end in 0?
6. Are the sets of numbers in Exercise 5 closed with respect to multiplication?
7. Are all sets of counting numbers which are closed under addition also closed under multiplication? Why?
8. Are any of the sets of numbers in Exercise 5 closed under subtraction?
9. Are any of the sets of numbers in Exercise 5 closed under division?

Exercises 3-5-b

Practice in Arithmetic Processes

1. Add: a. $\begin{array}{r}476\\398\\7256\\89\\305\\\underline{54}\end{array}$ b. $\begin{array}{r}403\\213\\414\\898\\777\\\underline{460}\end{array}$ 2. Subtract: $\begin{array}{r}40302\\\underline{20305}\end{array}$

3. Multiply: $\begin{array}{r}728\\ \underline{304}\end{array}$

4. $37\overline{)15059}$

5. Add: $\begin{array}{r}7\frac{1}{2}\\ 2\frac{1}{4}\\ 9\frac{1}{4}\\ \underline{8\frac{3}{4}}\end{array}$

6. a. $7\overline{)32172}$
   b. $8\overline{)11760}$
   c. $4\overline{)20324}$

7. Add: $\begin{array}{r}\$7.50\\ \$8.03\\ \$\underline{4.57}\end{array}$

8. Subtract \$593.67 from \$983.04.

9. Round 4795 to the nearest hundred.

10. Write in words: 2,070,351.

11. If 8 oranges cost 48 cents, what is the cost of a dozen at that rate?

12. Multiply: a. $\begin{array}{r}9816\\ \underline{8}\end{array}$ b. $\begin{array}{r}50106\\ \underline{9}\end{array}$ c. $\begin{array}{r}357082\\ \underline{7}\end{array}$

3-6. Inverse Operations

Often we do something and then we undo it. We open the door; we shut the door. We open the window; we close the window. One operation is the inverse of the other.

The inverse of putting on your coat is taking off your coat. The inverse operation of divison is multiplication. The inverse operation of addition is subtraction.

Suppose you have \$220 in the bank and you add \$10 to it. Then you have \$220 + \$10 = \$230. Now undo this by drawing out \$10. The amount that remains is \$230 - \$10 = \$220. The athletic fund at your school might have \$1800 in the bank and after a game have \$300 more. Then the fund has \$1800 + \$300 or \$2100 in it. But the team needs new uniforms which cost \$300 so \$300 is withdrawn to pay for them. The amount left is \$2100 - \$300, or \$1800. These operations undo each other. Subtraction is the inverse of addition.

Of course, we could express this idea in more general terms. Let x represent the number of dollars originally in the bank. If the amount we deposit is b, then $x + b = a$, where a represents the number of dollars we now have in the bank. How shall we undo this operation? From the number of dollars represented by a, we subtract the number of dollars withdrawn, represented by b, and we have the number represented by x. We write $x = a - b$.

You use the idea of inverse operation when you use addition in checking subtraction. For example:

$$\begin{array}{r} 203 \\ -\ 96 \\ \hline 107 \end{array} \qquad \begin{array}{r} a \\ -\ b \\ \hline x \end{array} \qquad \text{check:} \quad \begin{array}{r} 107 \\ +\ 96 \\ \hline 203 \end{array} \qquad \begin{array}{r} x \\ +\ b \\ \hline a \end{array}$$

You also use the idea of inverse operation when you use multiplication to check division. For example:

$$\begin{array}{r} 18 \\ 16\overline{)288} \\ \underline{160} \\ 128 \\ \underline{128} \\ 0 \end{array} \qquad \text{check:} \quad \begin{array}{r} 16 \\ \times\ 18 \\ \hline 128 \\ \underline{160} \\ 288 \end{array}$$

or $288 \div 16 = 18$ check: $288 = 18 \times 16$.

Notice that if a and b are whole numbers, and if $a > b$, then there is a whole number x so that $b + x = a$. Examples: If a is 17 and b is 10, then x is the whole number 7 so that $10 + 7 = 17$; if a is 41 and b is 35, then x is the whole number 6 so that $35 + 6 = 41$. When a is greater than b it is always possible to find x so that $a = b + x$. Can you make the same generalization if the above operation $b + x = a$, is changed to multiplication, $b \cdot x = a$? If you substitute 2 for b and 3 for a you will see that there is no whole number that can be substituted for x such that $2 \cdot x = 3$. If one substitutes certain numbers--for example, if $a = 20$ and $b = 4$--then there is a whole number that can be substituted for x such that $4 \cdot x = 20$. In this example x must represent 5, since $4 \cdot 5 = 20$. We get the 5 by dividing 20 by 4. Also:

If b is 6 and a is 24 then x must be 4 since $6 \cdot 4 = 24$.

If b is 5 and a is 40 then x must be 8 since $5 \cdot 8 = 40$.

If b is 3 and a is 30 then x must be 10 since $3 \cdot 10 = 30$.

In each example the number for x is found by dividing the number represented by a by the number represented by b. In general, if there is a counting number x that can be multiplied by a counting number b to get counting number a, then this number x can be found by dividing a by b. We write this as $b \cdot x = a$. We multiply x by b to obtain a. To undo the operation we must perform the inverse operation which means that we must divide a by b to obtain x: $b\overset{x}{\overline{)a}}$. The inverse operation of multiplying by b is dividing by b.

Exercises 3-6

1. Select the words or phrases that describe operations that have an inverse. An operation followed by its inverse returns to the original situation.
    a. Picking up the pencil. (Remember, "not picking up the pencil" is not an inverse operation. "Not picking up the pencil" does not undo the operation of picking up the pencil.)
    b. Put on your hat.
    c. Getting into a car.
    d. Extend your hand.
    e. Multiply.
    f. Build.
    g. Smell the rose.
    h. Step forward.
    i. Jump from a flying airplane.
    j. Addition.
    k. Cutting off a dog's tail.
    l. Subtraction.
    m. Looking at the stars.
    n. Talking.
    o. Taking a tire off a car.
2. Write the inverse operation to each of those operations selected in Exercise 1.

3. Perform the indicated operation and check by the inverse operation.

Subtract in (a) to (f)

a. $\begin{array}{r}89231\\ \underline{42760}\end{array}$

b. $\begin{array}{r}\$805.06\\ \underline{\$297.96}\end{array}$

c. $\begin{array}{r}803\text{ ft.}\\ \underline{297\text{ ft.}}\end{array}$

d. $\begin{array}{r}\$4302.14\\ \underline{\$2889.36}\end{array}$

e. $\begin{array}{r}\$8000.02\\ \underline{\$6898.98}\end{array}$

f. $\begin{array}{r}\$10040.50\\ \underline{\$\ 8697.83}\end{array}$

g. $29\overline{)25404}$

h. $38\overline{)37506}$

i. $27\overline{)21546}$

j. $19\overline{)13243}$

k. One hundred twenty minus eighty-seven.

l. The sum of six hundred forty-seven and eight hundred twenty-nine.

m. The difference between eighty-nine and twenty-one.

n. Seventy-six plus sixty-seven.

o. The product of three hundred six and one hundred ninety.

4. Find, if possible, a whole number which can be used for x in each of the following to make it a true statement. If there is no whole number that can be used for x, then say there is none.

a. $9 + x = 14$

b. $x + 9 = 14$

c. $x + 1 = 2$

d. $4 + x = 11$

e. $10 + x = 7$

f. $5 + x = 5$

g. $10 = x + 2$

h. $x = 9 - 5$

i. $x = 11 - 8$

j. $8 + x = 11$

k. $6 + x = 3$

l. $x = 13 - 6$

m. $3 + x = x + 3$

n. $3 \cdot x = 12$

o. $4 \cdot x = 20$

p. $x = 20 \div 4$

q. $2 \cdot x = 18$

r. $x = 18 \div 2$

s. $5 \cdot x = 30$

t. $2 \cdot x = 0$

u. $x = 0 \div 2$

v. $9 \cdot x = 0$

w. $x = 0 \div 9$

x. $3 \cdot x = 3$

y. $x = 3 \div 3$

z. $11 \cdot x = 11$

5. a. If one bookcase will hold 128 books and another 109 books, how many more books does the former hold?
   b. A theatre sold 4789 tickets one month and 6781 tickets the next month. How many more people came to the theatre the second month than came the first month?
   c. If one building has 900 windows and another building 811 windows, how many more windows does the first building contain?
   d. The population of a town was 19,891 people. Five years later the population was 39,110 people. What was the increase of population for the five years?
   e. If one truck can carry 2099 boxes, how many boxes can 79 similar trucks carry?
   f. How many racks are needed to store 208 chairs, if each rack holds 16 chairs?
   g. At a party there were 288 pieces of candy. If there were 48 children at the party, how many pieces of candy could each have?
   h. A girl scout troop has 29 members. Each member is to sell boxes of cookies. If the troop has 580 boxes to sell, how many boxes will each girl have to sell in order to sell all of them?

6. Perform the following operations:
   a. Add 16 and 17. From the sum subtract 12.
   b. Subtract 24 from 89. To this difference add 19.
   c. Multiply 27 by 34. Divide the product by 9 and then add 100.
   d. Find the sum of 9, 9 and 9. From it subtract 4, six times.
   e. Take 308, divide it by 28. Multiply the quotient by 5. Subtract 9 from the product.
   f. Find the difference between 47 and 38. Divide this difference by 3 and then add 17.
   g. Divide 272 by 16, multiply the quotient by 12 and subtract 100 from the product.

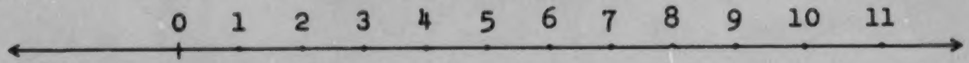
h. Multiply 12 and 13 and add 39. Divide the sum by the product of 3 and 13.

i. Add 26 and 42 and divide the sum by 17. To this add 117 and divide this sum by 11.

j. Find the difference between 87 and 49. Multiply this difference by 10 and subtract 40. Divide this last difference by 68 and then add 6.

3-7. Betweenness and the Number Line

How whole numbers are related may be shown with a picture. Select some point on a line as below and label it zero (0). Label the first dot to the right of zero the first counting number and each dot after that to the right the succeeding counting numbers. This picture is often referred to as The Number Line. Any whole number is smaller than any of the numbers on the right side of it and greater than any of the numbers on its left. For example 3 is less than 5 and greater than 2. This may be written $2 < 3 < 5$, since 2 is less than 3 and 3 is less than 5. With the number line we can also determine how many whole numbers there are between any two whole numbers. For example, to find how many whole numbers there are between 6 and 11 we can look at the picture and count them. We see four of them, 7, 8, 9, and 10.



Exercises 3-7

1. How many whole numbers are there between:
   - a. 7 and 25
   - b. 3 and 25
   - c. 20 and 25
   - d. 17 and 25
   - e. 25 and 25
   - f. 28 and 25
   - g. 26 and 25
   - h. 114 and 25

*1. If a and b are whole numbers, and $a > b$, is the number of whole numbers between a and b:
   (1) $b - a$?
   (2) $(a - 1) + b$?
   (3) $a - (b + 1)$?
   (4) $(a - b) + 1$?

2. What is the whole number midway between:
   a. 7 and 13
   b. 9 and 13
   c. 20 and 28
   d. 10 and 50
   e. 17 and 19
   f. 17 and 27
   g. 12 and 20
   h. 12 and 6

3. Which of the following pairs of whole numbers have a whole number midway between them?
   a. 6, 8
   b. 6, 10
   c. 8, 18
   d. 8, 13
   e. 7, 12
   f. 26, 33
   g. 9, 17
   h. 19, 36
   i. a, b if a and b are even whole numbers
   j. a, b if a and b are odd whole numbers
   k. a, b if a is odd and b is even.

4. The whole numbers a, b and c are so located on The Number Line that b is between a and c, and $c > b$.
   a. Is $c > a$? Explain with a number line.
   b. Is $b > a$? Explain with a number line.
   c. Is $b < c$? Explain with words.

5. The whole numbers a, b, c and d are so located on The Number Line that b is between a and c and a is between b and d. What relation, if any, is there among b, c, and d?

3-8. The Number One

The number one is a special number in several ways. One is the smallest of our counting numbers. We may build any number, no matter how large, by beginning with 1 and adding 1's until we have reached the desired number. For example, to obtain the number five, we can begin with our special number 1 and repeat the addition of 1. $1 + 1 = 2$, $2 + 1 = 3$, $3 + 1 = 4$, $4 + 1 = 5$. There is no largest counting number.

Also, it will be observed that for any of the counting numbers (1, 2, 3, ...) which we may select, we get the next larger counting number by adding 1. This may seem obvious to you because you have used the numbers so many times. In some of the fundamental operations we do not get the next counting number by operations using only the number 1; e.g., $3 \cdot 1 = 3$, $3 - 1 = 2$. In one case we do not even get a counting number. Observe what happens when we use the operation of subtraction: $1 - 1 = 0$. Zero is not a counting number.

In multiplication if we wish to obtain a different numeral for a number, we can multiply by a selected form of the special number 1. In this way we may get a different numeral, but it represents the same number. You may recall that in rewriting 4 as $\frac{8}{2}$, you were simply multiplying 4 by $\frac{2}{2}$. Of course, $\frac{2}{2}$ is our special number 1. Multiply $\frac{1}{3}$ by $\frac{3}{3}$ and get $\frac{3}{9}$; multiply $\frac{4}{5}$ by $\frac{2}{2}$ and get $\frac{8}{10}$. These are examples of multiplying by the number 1 in selected forms $\frac{3}{3}$, and $\frac{2}{2}$. This means that the new fractions are different in form from the original ones but they still represent the same number. The special number one when used as a multiplier makes the product identical with the multiplicand. Because the product of any counting number and one is the original counting number, the number 1 is called the "identity element" for multiplication.

Since division is the inverse operation of multiplication, is the number one also special in division? What happens if we divide any counting number by one? We do obtain the same counting number. But if we divide 1 by a counting number we do not get the counting number. For this reason we cannot say that the number one is the identity element for division. A counting number multiplied by 1 is the same number as 1 multiplied by the counting number but the same thing cannot be said for division. If we let

C represent any counting number we can express these multiplication and division operations using the number 1 in the following ways.

$$C \cdot 1 = 1 \cdot C;$$
$$C \div 1 = C;$$
$$C \div C = 1;$$
$$1 \div C \neq C \text{ if } C \neq 1.$$

We have learned to use 10^2 to mean $10 \cdot 10$; 10^3 to mean $10 \cdot 10 \cdot 10$; 10^6 to mean $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$. The "2," "3," "6" are called exponents. The exponents are small, but the numbers represented by 10^2, 10^3 and 10^6 are very large. If we use 1 in place of 10 this is not true. For $1^2 = 1 \cdot 1$; $1^3 = 1 \cdot 1 \cdot 1$; $1^6 = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$ and these are still the number 1. In fact 1^2 or 1^{200} or 1^{3056} is still 1.

When we say 1^2 or 1^{200} is really only 1, we are just giving different names to the same thing. It is true that the number represented by each of these expressions is 1. Can you think of other such combinations of symbols which represent 1? What number is represented by $5 - 4$? X - IX?

Our discussion of the number one may be summarized briefly in the mathematical sentences below. Can you translate them into words? The letter C here represents any counting number.

a. $C = 1$ or $(1 + 1)$ or $(1 + 1 + 1)$ or ... etc.
b. $1 \cdot C = C$
c. $C \div 1 = C$
d. $C \div C = 1$
e. $1^C = 1$.

Exercises 3-8

1. From the following symbols, select those that represent the number 1:

a. I
b. $\frac{4}{4}$
c. $5 - 4$
d. $1 - 0$
e. $1 + 0$
f. $1 \cdot 2$

[sec. 3-8]

g. $\frac{6}{4}$

h. $\frac{4}{1}$

i. 1

j. $1 \cdot 0$

k. $\frac{200}{200}$

l. 1^{10}

m. $\frac{1}{2} + \frac{1}{2}$

n. $1 \cdot 100$

o. $\frac{8 - 1}{12 - 5}$

p. $\frac{2 - 1}{1}$

2. Copy and fill in the blanks:

a. $100 \cdot 1 =$ _____

b. $10 \cdot 1 \cdot 1 \cdot 1 =$ _____

c. $\frac{14}{1} =$ _____

d. $1 \cdot \frac{2}{3} \cdot 1 =$ _____

e. $0 \cdot 1 =$ _____

f. $1 \cdot 0 =$ _____

3. Can you get any counting number by the repeated addition or subtraction of 1 to or from any other counting number? Give an example to support your conclusion.

4. By the above process can you get a number that is not a counting number? Give an example to support your conclusion.

5. Robert said, "The counting numbers are not closed under the subtraction of ones but they are closed under the addition of ones." Show by an example what Robert meant.

6. Perform the indicated operations:

a. $(4 - 3)\overline{)876429}$

b. $1\overline{)976538}$

c. $897638 \cdot (5 - 4)$

d. $896758 \cdot \frac{4}{4}$

e. $3479 \cdot 1^{110}$

f. $97 \cdot x^6$ (if x is 1)

g. $1^7 \cdot (489 \div 489)$

h. $\frac{8}{8} \cdot 1^5 \div 1^4$

3-9. The Number Zero

Another special number is zero. Occasionally you will hear it called by other names, such as "naught." When you answer a telephone a voice may say, "Is this 'one eight oh three'?" Of course, the caller is not referring to the letter "o," and all of us understand that he means "one eight zero three."

Although zero is not included in the counting numbers, it is considered as one of the whole numbers. Most of the time we use it according to rules of the counting numbers, and in a sense it is used to count. If you withdraw all your money from the bank, you can express your bank balance with this special number zero. If you have answered no questions correctly, your test score may be zero. If there are no chalkboard erasers in the classroom, the number of erasers may be expressed by zero. In all these cases, no money in the bank, no correctly answered questions and no erasers, the zero indicated that there are no objects or elements in the set of objects being discussed. If there are no elements in the set, we call it an empty set.

The number zero is the number of elements in the empty set. In this sense, some persons say that zero means "not any." Others say it means "nothing" because there is nothing in the set. As we shall see, these are rather confused and limited concepts of zero.

On a very cold morning Paul was asked the temperature. After looking at the thermometer he replied, "zero." Did he mean there was "not any"? Did he mean "nothing"? No, he meant the top of the mercury was at a specific point on the scale called zero. Fred had an altimeter in his car so he could check the altitude as they drove in the Rocky Mountains. On one vacation trip they drove to the Salton Sea. On the way down Fred exclaimed, "Look, the altitude is zero!" When the altimeter indicates zero, it does not mean there is "nothing," it means we are at a specific altitude which is called zero. It is just as specific and real as an altitude of 999 feet.

We noticed that the sum of a counting number and one is always the next larger counting number. The sum of a counting number and zero is always the original counting number. For example, $4 + 0 = 4$. We might express this fact in symbols $C + 0 = C$ where C is any counting number. Or we might express the fact by saying that zero is the "identity element" for addition.

The difference between the same two natural numbers is the special number zero. For example, $4 - 4 = 0$. Did you notice that in this subtraction operation you do not get a counting number?

To put the idea in more elegant language, we would say that the set of counting numbers is not closed under subtraction.

Let us look at the special number zero under the operation of multiplication. What could $3 \cdot 0$ mean? We might think of the number of chairs in 3 rooms if each room contains zero chairs. Thus, any number of rooms containing zero chairs would have a total of zero chairs. We might express this idea in symbols by writing $C \cdot 0 = 0$, where C is any counting number.

The product $0 \cdot 3$ is even more difficult to explain. But we do know by the commutative property for multiplication that $3 \cdot 0 = 0 \cdot 3$. We have seen that $3 \cdot 0 = 0$. Therefore, we must have $0 \cdot 3 = 0$ as we wish the commutative property for multiplication to be true for all whole numbers. If a represents any whole number, we may express this by writing $a \cdot 0 = 0 \cdot a = 0$. If a is zero we must have $0 \cdot 0 = 0$.

There is a very important principle expressed in the above symbols, but it may not be seen at the first glance. Did you observe that if the product of two or more whole numbers is zero, then one of the numbers must be zero? For example, $4 \cdot 5 \cdot 0 = 0$. In mathematics you will use this fact frequently.

Let us see if zero follows the rules for division of counting numbers.

What could zero divided by 3 mean? If we have a room with zero chairs and divide the room into three parts, it could mean the number of chairs in each part of the room. With this meaning, $0 \div 3$ should be 0. If $3\overset{0}{\overline{)0}}$ then 0×3 should be zero, by the inverse operation. Does this agree with the definition of multiplication by zero?

Occasionally students forget that the division of zero by a counting number is always zero and never a counting number. For example, $\frac{0}{7} = 0$, $\frac{0}{7} \neq 7$.

If $\frac{0}{7} = 0$, what is $\frac{7}{0}$? Is $0\overline{)7}$ a counting number? Let us assume that $0\overline{)7}$ is equal to some number represented by N. This means that 7 is equal to zero times some number N. $(7 = 0 \times N)$. The product of any number by zero is zero, therefore, there is no number N that will equal $0\overline{)7}$. In more elegant language, we may say that $\frac{7}{0}$ is not the name of any counting number or zero. Therefore, we cannot perform this operation. We cannot divide a counting number by zero.

Could we divide zero by zero? In symbols the question is "$\frac{0}{0} = ?$". Or $0\overset{?}{\overline{)0}}$. If $0\overline{)0}$ equals some number n then by our definition of multiplication, $0 \times n = 0$. What numbers could replace n? Could n be 3? Of course, n could be any counting number or zero. Since $0\overline{)0}$ could be any whole number, the symbol $\frac{0}{0}$ has too many meanings. Therefore, we should remember that we cannot divide either a counting number or zero by zero.

Mary summarized the operations with the special number zero in these symbols. State them in words if u and w represent any whole numbers and C represents any counting number.

a. $w + 0 = w$
b. $0 + w = w$
c. $w - 0 = w$
d. $0 \cdot w = 0$
e. $w \cdot 0 = 0$
f. If $u \cdot w = 0$, then either u or w is zero or both are zero.
g. $0 \div C = 0$
h. $C \div 0$ has no meaning.

Exercises 3-9

1. Select the symbols that represent zero:

a. $1 + 0$

b. 0

c. $\frac{4}{0}$

d. $\frac{0}{4}$

e. $5 - 4$

f. $7 - 7$

g. $\frac{8}{0}$

h. $0 + 0$

i. $1 - 1$

j. $100 - 100$

k. $0 \cdot 4$

l. $4 \cdot 0$

m. $0 \cdot 0$

n. $\frac{0}{10}$

o. $\frac{4 - 4}{2}$

p. $\frac{4}{2 - 2}$

q. $\frac{1}{2} - \frac{1}{2}$

r. $14 \cdot 25$

s. $12 \cdot 0$

t. $0 + 12$

u. $2 \cdot (4 + 6 + 0)$

v. $(2 \cdot 4) \div 0$

w. $\frac{4}{4}$

x. $\frac{36}{9} - \frac{36}{12}$

2. Perform the indicated operations, if possible:

a. $376 \cdot 49$

b. $678 \cdot 946$

c. $8984 \div 62$

d. $9484 \div 62$

e. $87 \times \$419.98$

f. $69 \times \$876.49$

g. $\$989.26(2 - 2)$

h. $1 \times \$846.25$

i. $5 \times \$14.13$

j. $679 \cdot \frac{4}{4}$

k. $379(146.8 - 145.8)$

l. $(34.6 - 33.6) \times 897$

m. $\$397.16 \div (4 - 3)$

n. $\$897.40 \div (3 - 3)$

o. $(480 \div 24) \div 20$

p. $\$1846 \div (\frac{1}{2} + \frac{1}{2})$

q. $487.97 \times \frac{4}{4} \times 0$

r. $49 \cdot 0 \cdot 47 \cdot 97$

s. $\$97.86 \times 0 \times 0$

t. $(9 - 9) \cdot \frac{7 + 2}{8 + 1}$

u. $976 \cdot 1^6$

v. $1^{12} \times \$97.46$

3. Can you find an error in any of the following statements? a and b are whole numbers.

a. $4 \cdot 0 = 0$

b. $0 \cdot 4 = 0$

c. $2 \cdot 1 = 2$

d. $1 \cdot 2 = 2$

e. If $a \cdot b = 0$, a or $b = 0$
f. If $a \cdot b = 1$, a or $b = 1$
g. If $a \cdot b = 2$, a or $b = 2$
h. If $a \cdot b = 3$, a or $b = 3$
i. If $a \cdot b = C$, a or $b = C$

3-10. Summary

1. The set of numerals $\{1, 2, 3, 4, 5, \ldots\}$ is the set of symbols for the counting numbers.
2. The set of numerals $\{0, 1, 2, 3, 4, 5, \ldots\}$ is the set of symbols for the whole numbers.
3. The commutative property for addition: $a + b = b + a$, where a and b represent any whole numbers.
4. The commutative property for multiplication: $a \cdot b = b \cdot a$, where a and b represent any whole numbers.
5. The associative property for addition: $a + (b + c) = (a + b) + c$ where a, b, c represent any whole numbers.
6. The associative property for multiplication: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ where a, b, c represent any whole numbers.
7. The distributive property: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
   and $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$
   where a, b, c are any whole numbers.
8. New symbols: $\{$set of elements$\}$; $>$ is greater than; $<$ is less than; $\neq$ is not equal to.
9. Set and closure. A set is closed under an operation if the combination of any two elements of the set gives an element of the set. The set of counting numbers is closed under addition and multiplication but not under division or subtraction.
10. Inverse operations. Subtraction is the inverse of addition, but subtraction is not always possible in the set of whole numbers. Division is the inverse of multiplication, but division is not always possible in the set of whole numbers; that is, division of one whole number by another whole number does not always yield a whole number.

11. The number line and betweenness. Each whole number is associated with a point on the number line. There is not always a whole number between two whole numbers.
12. Special numbers: 0 and 1. Zero is the identity for addition; 1 is the identity for multiplication; multiplication by 0 does not have an inverse; division by 0 is not possible.

How Are You Doing?
(Some Review Questions, Chapters 1 - 3)

1. $(122)_{\text{three}} = (\quad)_{\text{ten}} = (\quad)_{\text{five}}$.
2. A girl went to the pantry with only a 5-cup and a 3-cup container to get 4 cups of flour. Can this be done if nothing but the flour container is used in addition to the two containers? If yes, how?
3. What is the probability of throwing a 2 with one throw of a die?
4. Write MCXI in Hindu-Arabic numerals.
5. What is another way of writing 176^7?
6. When a clerk counts out our change after we have made a purchase, does he usually practice addition or subtraction?
7. The base of the number system that has the easiest multiplication facts to learn is __?__.
8. $(2010)_{\text{three}}$ written in expanded notation would be: $(2 \times \_) + (0 \times \_) + (1 \times \_) + (0 \times 1)$
9. If we were to use base 31 numeration we would have _____ different symbols.
10. The value of the 4 in $(4512)_{\text{six}}$ is _____ times the value of the 4 in $(41)_{\text{twenty-seven}}$.
11. The set of whole numbers differs in what (if any) way from the set of counting numbers?
12. Is this statement true? "I can show that the set of whole numbers is closed with respect to the operation of subtraction if I can find one example such as 12 - 8 = 4 to illustrate this."

13. $(7 \cdot 3) + (3 \cdot 13) = (7 \cdot 3) + (13 \cdot 3)$. In the example shown, which property of the whole numbers is illustrated?
14. In the set of counting numbers the identity for multiplication is _____.
15. Zero is included in our set of whole numbers in order that _____? (State one of the special properties of zero.)
16. Use the distributive property and rewrite: $(2 \cdot 13) + (5 \cdot 13)$.
17. Use the associative property of addition so that the sum can be found easily: $136 + 25 + 75$.
18. Check by the inverse operation to see if $715 \div 11 = 65$.
19. The number of counting numbers between 6 and 47 is _____.
20. If we have the expression $(7 \cdot 3) + (6 \cdot 5) \cdot 3$ and apply the associative property of multiplication we will have _________.

Chapter 4

NON-METRIC GEOMETRY

This year you have been studying a great deal about numbers and their properties, but numbers are not the only things in mathematics that interest people. Living as we do in the "Space Age," we hear much about points, lines, planes, and space. The study of ideas like these is called geometry.

For over 4,000 years men have studied geometry in trying to understand better the world in which they lived. The geometry which we study is really the same as that studied by the Greeks over 2,000 years ago. A famous Greek mathematician named Euclid wrote many books about this geometry. Even today we call this part of mathematics "Euclidean" geometry. Our geometry is the same, but some of our words and ways of looking at things are quite different.

In this chapter we will use numbers for counting only. We do not use the idea of distance or measurement. This chapter is called "non-metric" geometry, but it might also be called "no-measurement" geometry. Do you see that non-metric means no-measurement?

Today when we are sending rockets toward the moon and satellites into orbit, the study of geometrical ideas is more important than ever before.

4-1. Points, Lines, and Space

Points

The idea of a point in geometry is suggested by the tip of a pencil or a dot on the chalkboard. A geometric point is thought of as being so small that it has no size. In geometry we do not give a definition for the term "point." What we do instead is to describe many properties of points. In this way we come to understand what mathematicians mean by the term "point."

Space

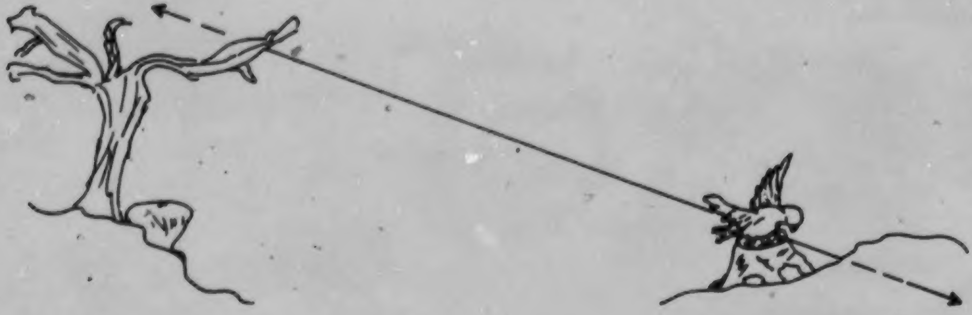
We think of space as being a set of points. There is an unlimited quantity of points in space. In a way, we think of the points of space as being described or determined by position--whether they are in this room, in the world, or in the universe.

Lines

For us, a line is a set of points in space, not any set of points but a particular type of set of points. The term "line" means "straight line." All lines in our geometry are understood to be straight. A line is suggested by the edge of a ruler. It is suggested by the intersection of a side wall and the front wall of your classroom.

A geometric line extends without limit in each of two directi
It does not stop at a point. The intersection of a side wall and the front wall of your classroom stops at the floor and the ceiling. The line suggested by that intersection extends both up and down, indefinitely far.

You have probably heard people say, "as the crow flies." A crow usually flies directly from one landing point to another. The expression therefore means "in a straight line." Crows do not flit about as bats do. The path of flight of a crow, then, suggest
a geometric line. We should understand that the "line" does not start or stop at the crow's resting places. It extends endlessly in both directions.



Think about two students holding a string stretched between them. In any position where it is straight, the taut string marks a portion of a line. It is the line through one student's fingers and also through the other's. The line itself goes beyond where they hold the string. The string is not the line or any part of a line. It just represents the line we know to be there.

With the students' fingers in the same positions, is there more than one possible position for the taut string? You probably think, "Of course not," and you are right. You now have discovered a basic property of space.

Property 1: Through any two different points in space there is exactly one line.

As you can see, there is an unlimited quantity of lines in space!

By using lines we can get a good idea of what space is like. Consider a point at a top corner of your teacher's desk. Now consider the set of all points suggested by the walls, the floor, and the ceiling of your classroom. Then for each point of this set there is a line through it and the selected point on your teacher's desk. Each line is a set of points. Space is made up of all the points on all such lines. Remember, these lines extend outside the room.

Just as we do not precisely define "point" and "line," we do not precisely define "space." We study its properties and in this way understand it. We recall that this idea of space is like that which was known to the ancient Greeks. It has been studied by students ever since then. We can really understand ideas like those of space and point only after a great deal of study. We can't expect to learn everything about them in a few days, a week, or even this year.

Class Discussion Problems

1. Consider one of the lines that pass through the pencil sharpener and the knob of the entrance door to your classroom. What objects in the room are "pierced" by this line? What objects

outside the room are also "pierced" by this line?

2. A mathematical poet might say, "Space is like the bristling, spiny porcupine." In what ways is this description like that in the discussion following the statement of Property 1?

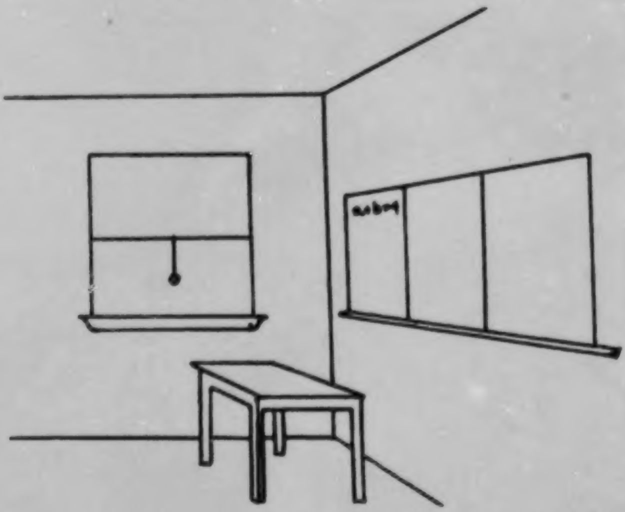
Exercises 4-1

1. For party decorations, crepe-paper ribbons were draped between two points on the gymnasium walls. Does this show, contrary to Property 1, that there may be several geometric lines through two points?
2. When a surveyor marks the boundaries of a piece of land, he places stone "monuments" (small stone blocks) at the corners. A small hole or nail in the top of each monument represents a point. He knows that, if the monuments are not moved out of position, the original boundaries may be determined at any later time. How can he be sure of this?
3. A harp or violin player must learn exactly where to place his fingers on the strings of his instrument to produce the sounds he wishes to hear. Sometimes a string will break and be replaced by a new one. How does he know, without looking, where to place his fingers so that they will rest on the new string?

4-2. Planes

Any flat surface such as the wall of a room, the floor, the top of a desk, a piece of plywood, or a door in any position suggests the idea of a plane in mathematics. Like a line in mathematics, a plane is thought of as being unlimited in extent. If you begin at a starting point on a plane and follow paths on the plane in all possible directions, these paths will remain in the plane but have no endings. A plane, then, would have no boundaries!

We think of a plane as containing many points and many lines. As you look at the wall you can think of many points on it, and you can also think of the lines containing these points. The edge where the side wall of a room meets the ceiling suggests a line in either the plane represented by the ceiling or the plane represented by the side wall. The edges of the chalk tray represent lines in the classroom. At least one of these is in the plane represented by the chalkboard. Any number of points and lines could be marked on the chalkboard to represent points and lines in geometry.



Mathematicians think of a plane as a set of points in space. It is not just any set of points, but a particular kind of set. We have already seen that a line is a set of points in space, a particular kind of set and a different kind from that of a plane. A plane is a mathematician's way of thinking about the "ideal" of a flat surface.

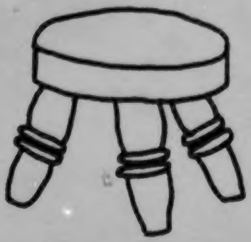
If two points are marked on the chalkboard, exactly one line can be drawn through these points. Is this just what Property 1 says? This line is on the chalkboard. The plane represented by the chalkboard contains the set of points represented by the line which you have drawn.

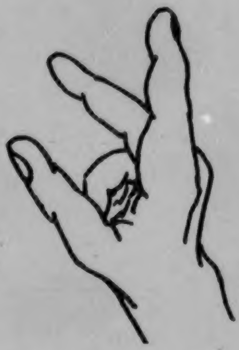
Think of two points marked on a piece of plywood. Part of the line through these points can be drawn on the plywood (recall that "line" means "straight line"). Must the line through these two points be on the plane of the plywood? We can now conclude:

Property 2: If a line contains two different points of a plane, it lies in the plane.

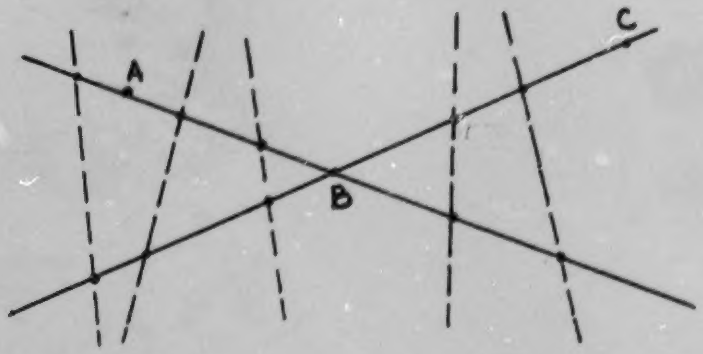
Notice a pair of corners of the ceiling in the front of your room. In how many planes is this pair of points contained? If one thinks of the ends of the binding of his tablet as a pair of points, he can see that the planes represented by the pages of the tablet contain these points.

The question might be asked, "How many planes contain a specified pair of points?" The tablet with its sheets spread apart suggests that there are many planes through a specified pair of points. The front wall and the ceiling represent two planes through the point represented by the two upper front corners of the room. Show by means of hand motions the positions of other planes through these two points. A door in several positions represents many planes passing through a pair of points--its hinges. We can say, "Many planes contain a pair of points."





Suppose next that we have three points not all on the same line. Three corners of the top of your teacher's desk is an example of this. The bottoms of the legs of a three-legged stool is another example. Such a stool will stand solidly against the floor, while a four-legged chair does not always do this unless it is very well constructed. Spread out the thumb and first two fingers of your right hand as in the figure on the previous page. Hold them stiffly and think about their tips as being points. Now take a book or other flat surface and attempt to place it so that it lies on the tips of your thumb and two fingers (on the three points). Can you hold the book against the tips of your thumb and two fingers? If you bend your wrist and change the position of your thumb and fingers, will the book be in a new position? With your hand in any one position, is there more than one way in which a flat surface can be held against the tips of your fingers? It seems clear that there is only one position for the flat surface.



Property 3: Any three points not on the same line are in only one plane.

Do you see why this property suggests that if the legs of a chair are not exactly the same length that you may be able to rest the chair on three legs, but not on four?

In the figure, there are three points, A, B, and C in a plane. The line through points A and B and the line through points B and C are drawn. The dotted lines are drawn so that they contain two points of the plane of A, B, and C. Each dotted line contains a point of the line through A and B, and a point of the line through B and C. The dotted lines are contained in the plane. Which property says this? The sets of points represented by the dotted lines are contained in the plane. The plane which contains A, B, and C can now be described. It is the set of all points which are on lines containing two points of the figure consisting of the lines through A and B and through B and C.

Class Discussion Problems

1. A plane contains three points suggested by the two front feet of the teacher's desk and the pencil sharpener. Through what objects in the room does this plane pass? What objects outside of the room?

2. Place a ruler, edge up, upon your desk. Keep one edge of a surface, such as a piece of cardboard, in moving contact with the desk top but at the same time keep the surface in contact with the upper edge of the ruler. Make the surface slope gradually, then steeply. In how many different sloping positions may the "surface" be placed? On your desk, near but not in line with the ruler, place an eraser, tack, marble, or some other small object that would suggest a point. Place the surface on this object and the upper edge of the ruler. In how many different sloping positions may the surface be placed this time? How do these two situations show what is meant by Property 2 and Property 3?

Exercises 4-2

1. In a certain arrangement of three different points in space, the points can be found together in each and every one of many planes. How are the points arranged?
2. In another arrangement, three different points can be found together in only one plane. How are the points arranged?
3. Photographers, surveyors, and artists generally use tripods to support their equipment. Why is a three-legged device a better choice for this purpose than one with four legs? How does Property 3 apply here?

4. How many different lines may contain:
   a. One certain point?
   b. A certain pair of points?
5. How many different planes may contain:
   a. One certain point?
   b. A certain pair of points?
   c. A certain set of three points?
6. Why is the word "plane" used in the following names: airplane, aquaplane? Find out what the dictionary states about plane sailing, plane table, planography.

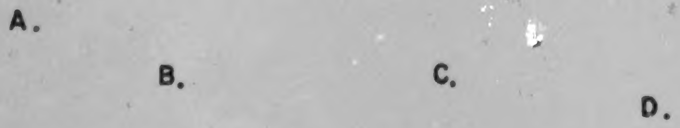
*7. How many lines can be drawn through four points, a pair of them at a time, if the points lie:
   a. In the same plane?
   b. Not in the same plane?

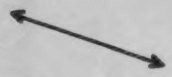
8. BRAINBUSTER. Explain the following: if two different lines intersect, one and only one plane contains both lines.

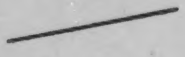
4-3. Names and Symbols

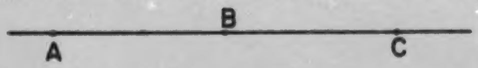
It is customary to assign a letter to a point and thereafter to say "point A" or "point B" according to the letter assigned. Short-cut arrangements like this are good, but we should be certain that their meanings are clearly understood.

A dot represents a point. We shall tell which point we have in mind by placing a letter (usually a capital) as near it as possible. In the figure below, which point is nearest the left margin? Which point is nearest the right margin?



A line may be represented in two ways, like this 

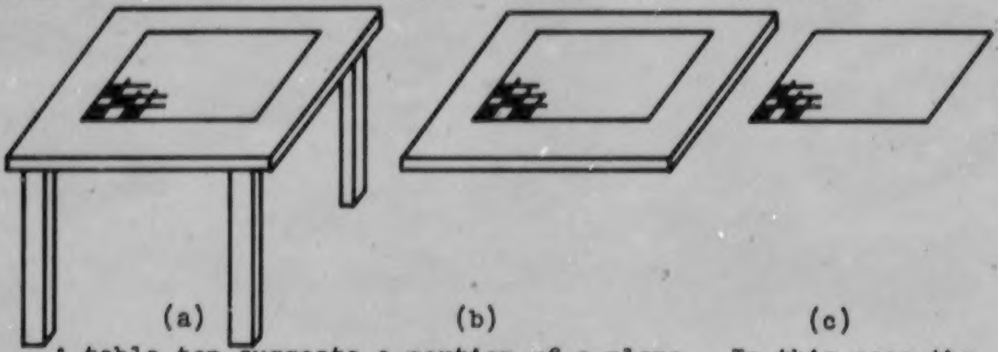
or simply like this  . In the first drawing, what do the arrowheads suggest? Does a line extend indefinitely in two directions? The drawing, then, suggests all the points of the line, not just those that can be indicated on the page.

If we wish to call attention to several points on a line, we shall do it in this way: 

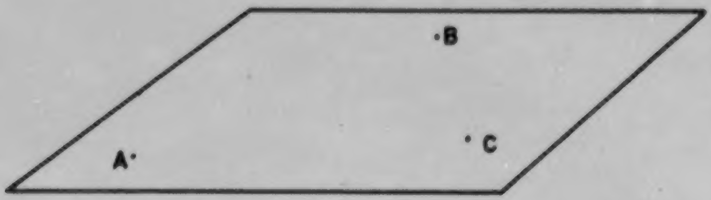
and the line may be called "line AB." A symbol for this same line is $\overleftrightarrow{AB}$. Other names for the above line are "line AC" or "line BC." The corresponding symbols would be $\overleftrightarrow{AC}$ or $\overleftrightarrow{BC}$.

Notice how frequently the word "represent" appears in these explanations. A point is merely "represented" by a dot because as long as the dot mark is visible, it has size. But a point, in geometry, has no size. Also lines drawn with chalk are rather wide, wavy, and generally irregular. Are actual geometric lines like this? Recall that "line" for us means "straight line." A drawing of a line by a very sharp pencil on very smooth paper is more like our idea of a line, but its imperfections will appear under a magnifying glass. Thus, by a dot we merely indicate the position of a point. A drawing of a line merely represents the line. The drawing is not the actual line. It is not wrong to draw a line free-hand (without a ruler or straight edge) but we should be reasonably careful in doing so.

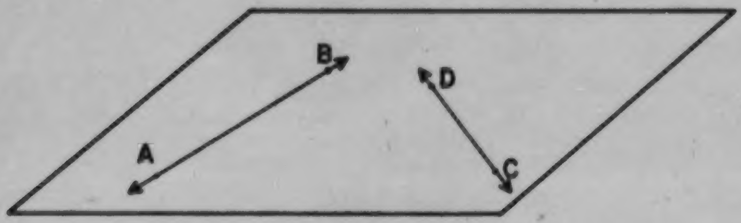
Just as we need to represent point and line, we find it necessary to "represent" a plane. Figure (a) is a picture of a checkerboard resting on top of a card table; figure (b) the same with the legs removed and figure (c) the checkerboard alone.


(a) (b) (c)

A table top suggests a portion of a plane. In this case the checkerboard suggests a smaller portion and the individual sections of the checkerboard still smaller portions of the plane. If we want to make a drawing on a piece of paper, we use a diamond-shaped figure as above. Points of a plane are indicated in the same way as points of a line:

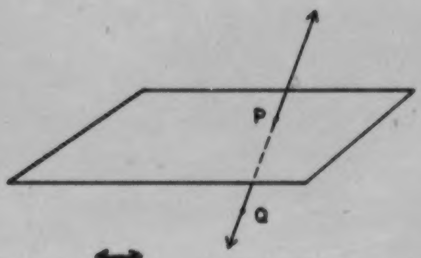


In the figure above, do points A and C seem closer to you than point B? If not, imagine point B as an opponent's checker at the far edge of the checkerboard. Then A and C would be checkers belonging to you.



In the above figure, A, B, C, and D are considered to be points in the plane of the checkerboard. A line is drawn througn point A and point B. Another line is drawn through point C and point D. According to Property 2, what can be said about $\overleftrightarrow{AB}$? Property 2 states, "If a line contains two points of a plane, it lies in the plane." What can be said about $\overleftrightarrow{DC}$?

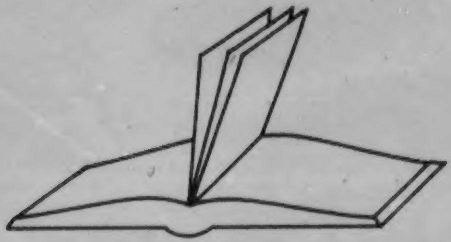
It is possible that a line might pierce or "puncture" a plane. A picture of this situation may appear thus:



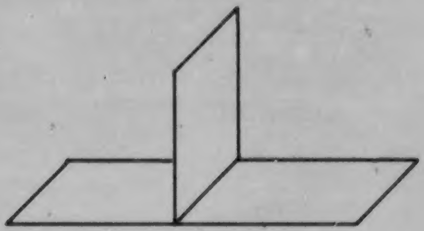
The dotted portion of $\overleftrightarrow{PQ}$ would be hidden from view if the part of the plane represented were the upper surface of some object such as the card table.

Once again, we see that the drawing only "represents" the situation.

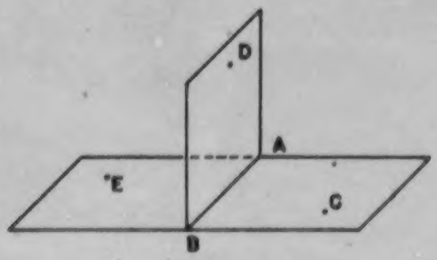
This is a drawing of an open book:



If all pages of the book except one of the upright set were removed, then the drawing would change to this:



This now suggests the intersection of two planes, one containing the front and back covers and the other containing the single page. Since there is more than one plane in the figure, it now becomes necessary to label them. Let us assign letters to some of the points of the diagram.



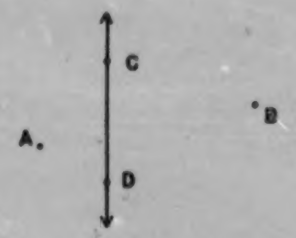
How could we use Property 3 to identify the planes of the figure above? Remember that Property 3 states, "Any three points not all on the same line are in exactly one plane." Would the letters ABD suggest the upright page or the book cover? Note that point A, point B, and point D are "not all on the same line." Many people would agree that the upright page is suggested by these letters. The page in question suggests a plane and may now be designated as "plane ABD." Following this same arrangement, the book cover may be called, "plane ABC," but it seems that "plane ABE" would be another way of indicating the book cover! To show that apparently two names for the same plane may be given, it might be possible to write plane ABC = plane ABE if we are certain of what the equal sign means as used here.

The notion of "set" will be helpful in explaining what is meant by "equal" when applied this way. Let's see what facts can be ascertained about the situation described by the figure:

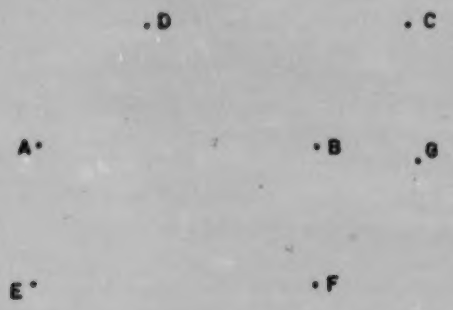
1. Plane ABC is a set of points extending beyond the book cover.
2. Plane ABE is a set of points extending beyond the book cover.
3. Points A, B, C, E, and others not indicated are in plane ABC and are also in plane ABE.

In other words, all elements of plane ABC, (a set of points) and elements of plane ABE (a set of points) seem to be contained in both sets (planes). We shall say, "Two sets are equal if and only if they contain the same elements." According to this, plane ABC = plane ABE. In other words, we say set M is equal to set N if M and N are two names for the same set.

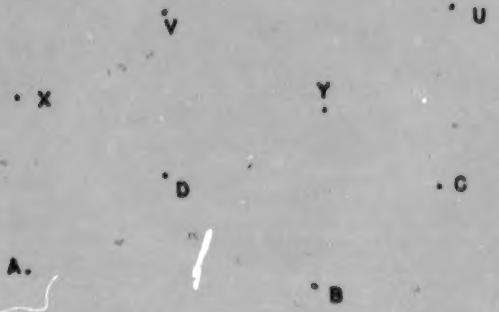
Exercises 4-3



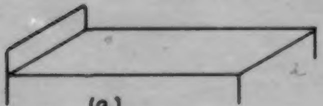
1. As one usually holds a page, which indicated point of the figure above is to the left of CD? To the right? Which indicated point is nearest the top of the page? The bottom?

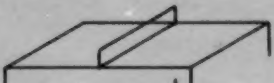


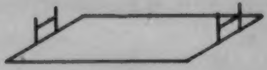
2. Transfer the points in the above figure to a piece of paper by tracing. From now on we shall refer to the copy you have made. With a portion of a line, join A to B, then B to C, C to D, D to A in that order. Now join A to E, B to F, C to G. What familiar piece of furniture might this sketch represent?

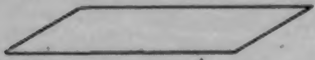


3. Make a tracing of the above figure. Join A to B, B to C, C to D, D to A, A to X, B to Y, D to V, and C to U. What has happened to the table?


(a)


(b)


(c)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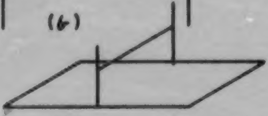
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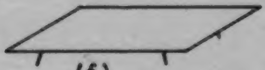
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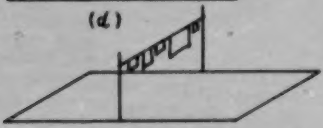
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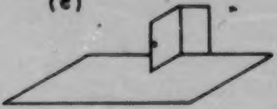
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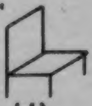
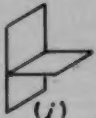
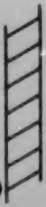

(i)


(j)


(k)

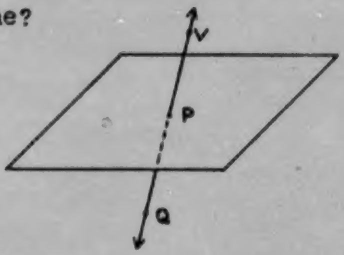
4. Each of the above sketches represents one of the familiar objects listed below. Match the sketches with the names.

Cot	Football field	Line of laundry
Ping Pong table	High jump	Ladder
Carpet	Coffee table	Open door
Chair		Shelf

5. Make sketches representing each of the following:

Desk	Pyramid with square base
Cube	Cereal box

6. In the figure below, is point V a point of $\overleftrightarrow{PQ}$? Is point Q an element of the plane? Is V? How many points of $\overleftrightarrow{PQ}$ are elements of the plane?



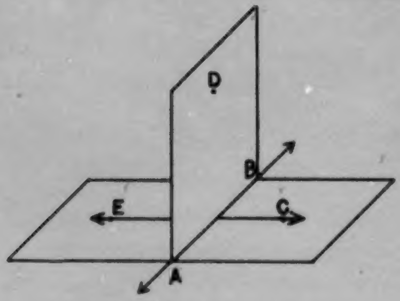


Figure (a)

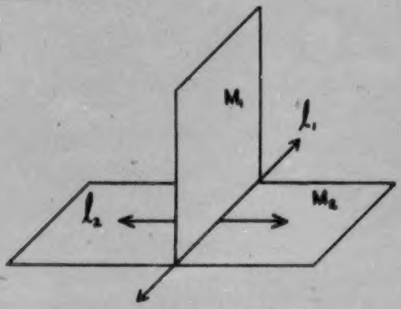


Figure (b)

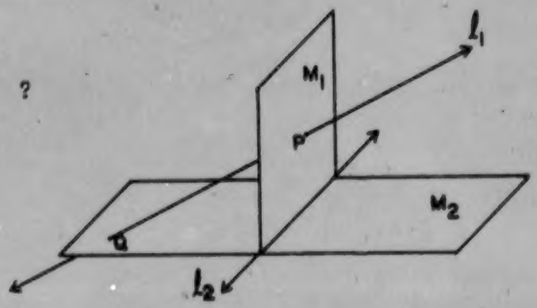
7. Figure (b) is a copy of figure (a), except for labeling. Three boys named "Tom," who are in the same class might be called T_1, T_2, and T_3 to avoid confusing one with the other. Similarly, three different lines may be denoted as ℓ_1, ℓ_2, and ℓ_3. The small numbers are not exponents. They are called "subscripts." Plane ABD in figure (a) corresponds to M_1 in figure (b). $\overleftrightarrow{AB}$ in figure (a) corresponds to ℓ_1 in figure (b).

In the left-hand column are listed parts of figure (a). Match these with parts of figure (b) listed in the right-hand column:

Parts of Figure (a)	Parts of Figure (b)*
1. $\overleftrightarrow{EC}$	a. ℓ_1
2. Plane ABC	b. ℓ_2
3. Plane ABD	c. M_1
4. Plane EBA	d. M_2
5. $\overleftrightarrow{AB}$	
6. The intersection of plane ABC and plane ABD	

*Does the second column suggest an advantage of the subscript way of labeling?

8. In the figure at the right, (a) does ℓ_1 pierce M_1? (b) Also M_2? (c) Is ℓ_1 the only line through P and Q? (d) What is the intersection of M_1 and M_2? (e) Is ℓ_1 in M_2? (f) Would ℓ_1 meet ℓ_2? (g) Are ℓ_1 and ℓ_2 in the same plane?



4-4. Intersection of Sets

We now shall introduce some useful and important ideas about sets.

Let set A = {1, 2, 3, 4, 5, 6, 7, 8, 9}
Let set B = {2, 4, 6, 8, 10, 12, 14, 16}

Let set C be the set of those elements which are in set A and are also in set B. We can write set C = {2, 4, 6, 8}. We call C the intersection of set A and set B.

Let set R be the set of pupils with red hair.
Let set S be the set of pupils who can swim.

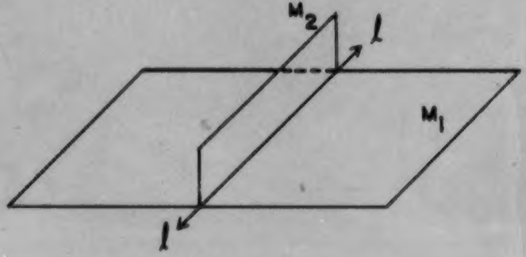
It might happen that an element of set R (a pupil with red hair) might be an element of set S (a pupil who can swim). In fact, there may be no such common elements or there may be several. In any case, the set of red-headed swimmers is the intersection of set S and set R.

A set with no elements in it is called the "empty set." Thus, if there are no red-headed swimmers, then the intersection of set S and set R is the empty set.

Let set H be the set of pupils in your classroom and let set K be the set of people under two years of age. Then the intersection of H and K is the empty set. There are no pupils in your classroom under two years of age!

We use the symbol $\cap$ to mean "intersection," that is, $E \cap F$ means "the intersection of set E and set F." Thus, referring to the sets mentioned previously, we write:

$A \cap B$ is {2, 4, 6, 8}
$R \cap S$ is the set of red-headed swimmers
$H \cap K$ is the empty set.



M_1 is like a ping-pong table. M_2 is like the net.

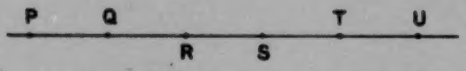
The figure above suggests two planes M_1 and M_2. The line ℓ seems to be in M_1 and also in M_2. Every point in M_1 which is also in M_2 seems to be on the line ℓ. Thus the following statement seems to be true: $M_1 \cap M_2 = \ell$.

Some people talk about intersections in a slightly different way than we have here. When we say the intersection of two sets is empty, they say that the two sets do not intersect. When we say the intersection of two sets is not empty, they say that the two sets do intersect. The ideas are the same but the language is a bit different.

Exercises 4-4

1. Write the set whose members are:
   a. The whole numbers greater than 17 and less than 23
   b. The cities over 100,000 in population in your state
   c. The members of the class less than 4 years old
2. Write three elements of each of the following sets:
   a. The odd whole numbers
   b. The whole numbers divisible by 5

c. The set of points on the line below, some of which are labeled in the figure:

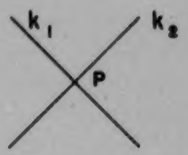


3. Give the elements of the intersections of the following pairs of sets:

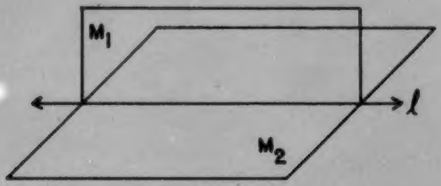
   a. The whole numbers 2 through 12 and the whole numbers 9 through 20

   b. The members of the class and the girls with blond hair

   c. The set of points on line k_1 and the set of points on line k_2



   d. The set of points on plane M_1 and the set of points on plane M_2



4. Let $S = \{4, 8, 10, 12, 15, 20, 23\}$
   $T = \{2, 7, 10, 13, 15, 21, 23\}$
   Find $S \cap T$.

5. Think of the top, bottom, and sides of a chalk box as sets of points.

   a. What is the intersection of two sides that meet?

   b. What is the intersection of the top and bottom?

6. Let S be the set of New England States, T be the set of states whose names begin with the word "New," and V be the set of states which border Mexico.

   a. List the states in the three sets, S, T, and V using the { } notation.

   b. What is $S \cap T$?

c. What is $S \cap V$?

d. What is $V \cap T$?

*7. The set of whole numbers which are multiples of 3 is closed under addition.

a. Is the set of whole numbers which are multiples of 5 closed under addition?

b. Is the intersection of the sets described in this exercise closed under addition? Why?

8. BRAINBUSTER. Explain why "intersection" has the closure property and is both commutative and associative. In other words, if X, Y, and Z are sets, explain why:

a. $X \cap Y$ is a set.

b. $X \cap Y = Y \cap X$.

c. $(X \cap Y) \cap Z = X \cap (Y \cap Z)$.

4-5. Intersections of Lines and Planes

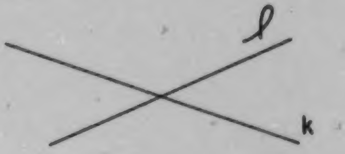
Two Lines

Look at a chalk box and think of the edges as representing lines. Some of these lines intersect and some do not. On the top of the box are some lines (edges) which intersect and some which do not. If we think of the lines contained in the top and the bottom of the box, we see intersecting pairs, pairs which do not intersect but have the same direction, and pairs which do not intersect and do not have the same direction. Is this situation also true of the edges in your classroom? (An edge is the line of intersection of two walls, a wall and the ceiling, or a wall and the floor.)

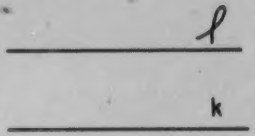
Can you hold your two arms in a position so that they represent intersecting lines? Can you hold them in a position so that the lines they represent do not intersect but have the same direction? Can you hold them in a position so that the lines they represent do not intersect and do not have the same direction? Are there any other possibilities as far as intersections of pairs of lines are concerned?

The possible intersections of two different lines may be described in three cases. Figures are drawn to represent the first two cases but not the third. As you read the third case, can you think of the reason it is difficult to represent it in a figure? Look at the figure for Problem 8 of Exercises 4-3. Could this be used for case three?

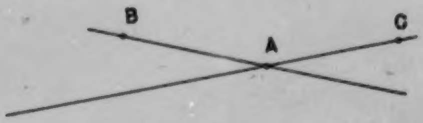
Case 1. ℓ and k intersect, or
$\ell \cap k$ is not the empty set.
ℓ and k cannot contain the same two points. Why?



Case 2. ℓ and k do not intersect and are in the same plane.
$\ell \cap k$ is the empty set, and ℓ and k are in the same plane. ℓ and k are said to be parallel.



Case 3. ℓ and k do not intersect and are not in the same plane.
$\ell \cap k$ is the empty set, and ℓ and k are not in the same plane. We say that ℓ and k are skew lines.

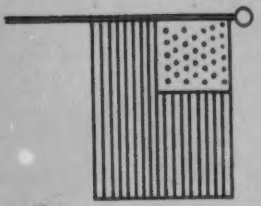


In the BRAINBUSTER of section 4-2 you were asked to explain why two lines lie in the same plane if they intersect. In the figure above are shown two lines which intersect in point A. B is a point on one of the lines and C a point on the other. By Property 3, there is exactly one plane which contains A, B, and C.

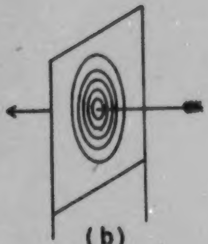
By Property 2, AB is in this plane.

By Property 2, AC is in this plane.

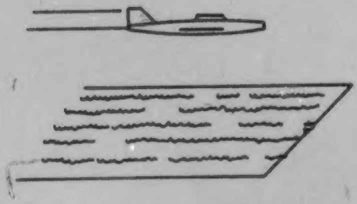
There is exactly one plane which contains the two lines.



(a)



(b)



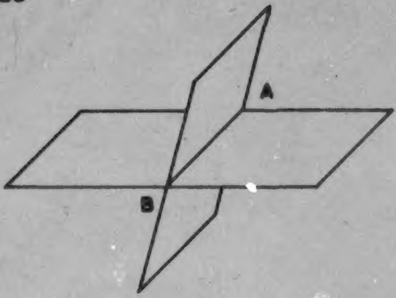
(c)

A Line and a Plane

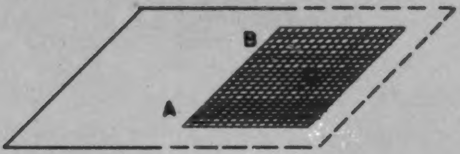
There is a way to arrange a line and a plane so that their intersection contains only one point. Does one of the drawings above suggest this way? There is another way to arrange a line and plane so that their intersection contains many points! Which drawing suggests this way? There is still another way to arrange a line and a plane so that their intersection contains no points at all! If the first two drawings were chosen correctly you won't have difficulty in choosing the correct drawing this time. If we refer to each of the above arrangements as a "case," then these three "cases" might be suggested also by the sides and edges of a chalk box, a shoe box, or the walls and edges of the room.

Two Planes

Next, let us think of two different planes in space. Suppose their intersection is not empty. Does the intersection contain more than one point? Notice that the planes of the front wall and of a side wall of a room intersect in more than one point. If you have two sheets of paper and you hold a sheet of paper in each hand, it might seem that the pieces or sheets of paper could be held so that they have only one point of intersection. But if we consider the planes of the sheets of paper and not just the sheets themselves, we see that if they have one point in common, it follows that their intersection will necessarily contain other points. Can you hold the two sheets so that they are flat and still represent planes that would intersect in only two points? Keeping the sheets as flat as possible, can you hold them so that they intersect in a curved line like the arch of a bridge?



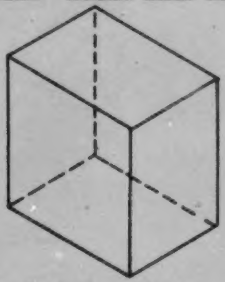
(d)



(e)

Let A and B be two points, each of which lies in two intersecting planes as in figure (d). From Property 2, the line $\overleftrightarrow{AB}$ must lie in each of the planes. Hence the intersection of the two planes contains a line. But if, as in figure (e), the intersection contains a point C not on the line $\overleftrightarrow{AB}$, then the two planes would be the same plane. By Property 3 there would be exactly one plane containing A, B, C. We now state:

Property 4: If the intersection of two different planes is not empty, then the intersection is a line.



If the intersection of two planes is the empty set, then the planes are said to be parallel. Several examples of pairs of parallel planes are represented by certain walls of a room or a stack of shelves. Can you think of others?

In discussing the intersection of two different planes we have considered two cases. Let M and N denote the two planes.

Case 1. $M \cap N$ is not empty. $M \cap N$ is a line.

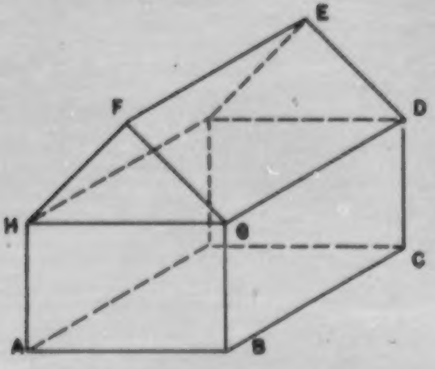
Case 2. $M \cap N$ is empty. M and N are parallel.

Are there any other cases? Why?

Exercises 4-5

1. List the three cases for the intersection of a line and a plane. Indicate whether the intersection contains 0, 1 or more than 1 point.
2. Describe two pairs of skew lines suggested by edges in your room.
3. On your paper, label three points A, B, and C so that $\overleftrightarrow{AB}$ is not $\overleftrightarrow{AC}$. Draw the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$. What is $\overleftrightarrow{AB} \cap \overleftrightarrow{AC}$?
4. Carefully fold a piece of paper in half. Does the fold suggest a line? Stand the folded paper up on a table (or desk) so that the fold does not touch the table. The paper makes sort of a tent. Do the table top and the folded paper suggest three planes? Is any point in all three planes? What is the intersection of all three planes? Are any two of the planes parallel?
5. Stand the folded paper up on a table with one end of the fold touching the table. Are three planes suggested? Is any point in all three planes? What is the intersection of the three planes?
6. Hold the folded paper so that just the fold is on the table top. Are three planes suggested? Is any point in all three planes? What is the intersection of the three planes?
7. In each of the situations of Exercises 4, 5, and 6, consider only the line of the fold and the plane of the table top. What are the intersections of this line and this plane? You should have three answers, one for each of 4, 5, and 6.
8. Consider three different lines in a plane. How many points would there be with each point on at least two of the lines? Draw four figures showing how 0, 1, 2, or 3 might have been your answer. Why couldn't your answer have been 4 points?

9. Consider this sketch of a house.

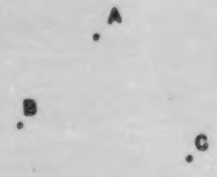


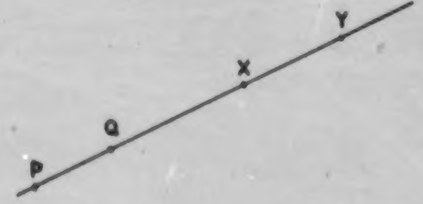
We have labeled eight points on the figure. Think of the lines and planes suggested by the figure. Name lines by a pair of points and planes by three points. Name:

a. A pair of parallel planes.
b. A pair of planes whose intersection is a line.
c. Three planes that intersect in a point.
d. Three planes that intersect in a line.
e. A line and a plane whose intersection is empty.
f. A pair of parallel lines.
g. A pair of skew lines.
h. Three lines that intersect in a point.
i. Four planes that have exactly one point in common.

4-6. Segments

Consider three points A, B, and C as in the figure below. Do we say that any one of them is between the other two? No, we usually do not.





We use the word "between" only when the points in question are on the same line (or on the same natural path). Look at points P, Q, and Y above. These points represent points on a line. Is X between Q and Y? Is Q between P and Y? Is P between X and Y? If you have said yes, yes, and no in that order, you are correct. All of us have a good natural sense of what it means to say that a point is between two other points on a line. We know that while Q is between P and X there are many other points between P and X, but we have not labeled any of these.

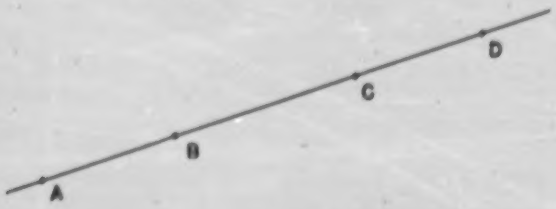
You will observe, when we say that a point P is between points A and B, that we mean two things: First, there is a line containing A, B, and P. Second, on that line, P is between A and B.

Let us look at the figure again. Is there some point between P and Q? We have not labeled any, but we understand there are many there. In fact, for any two points, A and B, in space we understand the situation is like that for P and Q. There is a line containing A and B, and on this line, there are points between A and B.

Finally, we are able to say what we mean by a segment. Think of two different points A and B. The set of points consisting of A, B, and all points between A and B is called the segment AB.

A and B are called the endpoints. We name the segment with endpoints A and B by $\overline{AB}$. Another name for this segment is $\overline{BA}$.

Every segment has exactly two endpoints. As suggested above, each segment contains points other than its endpoints. Sometimes a segment is called a line segment. It is not wrong to do so, but it is unnecessary. We understand, anyway, that a segment is a part of a line.



In the figure above, we can name segments $\overline{AB}$, $\overline{CB}$, and $\overline{CD}$. Are there other segments? Yes, there are segments $\overline{CA}$, $\overline{AD}$, and $\overline{BD}$. You recall that earlier we learned something about intersection of sets. What is the intersection of $\overline{AC}$ and $\overline{BD}$?

Not only may we talk about intersection of sets, but we also find it convenient to talk about the union of sets. The word "union" suggests uniting or combining two sets into a new set. The union of two sets consists of those objects which belong to at least one of the two sets. For example, in the figure above, the union of $\overline{AB}$ and $\overline{BC}$ consists of all points of $\overline{AB}$, together with all points of $\overline{BC}$, that is, the segment $\overline{AC}$.

We use the symbol $\cup$ to mean "union." That is, $X \cup Y$ means "the union of set X and set Y." Suppose that set X is the set of numbers {1, 2, 3, 4} and set Y is the set of numbers {2, 4, 6, 8, 10}. Do you have any idea of what $X \cup Y$ is? Yes, it is {1, 2, 3, 4, 6, 8, 10}. In the union of two sets we do not think of an element which occurs in both sets as appearing twice in the union. Instead, it appears just once.

Again, let us think of the set of all pupils who have red hair and the set of all pupils who can swim. We may think:

Let set R be the set of pupils with red hair.
Let set S be the set of pupils who can swim.

Then $R \cup S$ is the set of all pupils who either have red hair (whether or not they can swim) or who can swim (whether or not they have red hair).

Exercises 4-6

1. Draw a horizontal line. Label four points on it P, Q, R, and S in that order from left to right. Name two segments:
   a. Whose intersection is a segment.
   b. Whose intersection is a point.
   c. Whose intersection is empty.
   d. Whose union is not a segment.
2. Draw a line. Label three points of the line A, B, and C with B between A and C.
   a. What is $\overline{AB} \cap \overline{BC}$?
   b. What is $\overline{AC} \cap \overline{BC}$?
   c. What is $\overline{AB} \cup \overline{BC}$?
   d. What is $\overline{AB} \cup \overline{AC}$?
3. Draw a segment. Label its endpoints X and Y. Is there a pair of points of $\overline{XY}$ with Y between them? Is there a pair of points of $\overleftrightarrow{XY}$ with Y between them?
4. Draw two segments $\overline{AB}$ and $\overline{CD}$ for which $\overline{AB} \cap \overline{CD}$ is empty but $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$ is one point.
5. Draw two segments $\overline{PQ}$ and $\overline{RS}$ for which $\overline{PQ} \cap \overline{RS}$ is empty, but $\overleftrightarrow{PQ}$ is $\overleftrightarrow{RS}$.
6. Let A and B be two points. Is it true that there is exactly one segment containing A and B? Draw a figure explaining this problem and your answer.
7. Draw a vertical line ℓ. Label A and B two points to the right of ℓ. Label C a point to the left of ℓ. Is $\overline{AB} \cap \ell$ empty? Is $\overline{AC} \cap \ell$ empty?

8. BRAINBUSTER. In some old-fashioned geometry books the authors did not make any distinction between a line and a segment. They called each a "straight line." With "straight line" meaning either of these things, explain why we cannot say that "through any two points there is exactly one straight line."

9. BRAINBUSTER. Explain why "union" has the closure property and is both commutative and associative. In other words, if X, Y, and Z are sets, explain why:
    a. $X \cup Y$ is a set
    b. $X \cup Y = Y \cup X$
    c. $(X \cup Y) \cup Z = X \cup (Y \cup Z)$

*10. Show that for every set X, we have:

$$X \cup X = X$$

4-7. Separations

In this section we shall consider a very important idea--the idea of separation. We shall see this idea applied in three different cases.

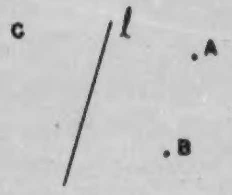
Let M be the name of the plane of the front chalkboard. This plane separates space into two sets: (1) the set of points on your side of the plane of the chalkboard, and (2) the set of points on the far side (beyond the chalkboard from you). These two sets are called half-spaces. The plane M is not in either half-space.

Let A and B be any two points of space not in the plane M of the chalkboard. Then A and B are on the same side of the plane M if the intersection of $\overline{AB}$ and M is empty, that is, if $\overline{AB} \cap M$ is empty. Also, A and B are on opposite sides of the plane M if the intersection of $\overline{AB}$ and M is not empty; in other words, there is a point of M between A and B.

Any plane M separates space into two half-spaces.

If A and B are in the same half-space, $\overline{AB} \cap M$ is empty. If A and B are in different half-spaces, $\overline{AB} \cap M$ is not empty. We call M the boundary of each of the half-spaces.

Now consider only the plane M of the front chalkboard. Do you see how the plane M itself could be separated into two half-planes? What could be the boundary of the two half-planes? Look at the next figure consisting of line, ℓ ,

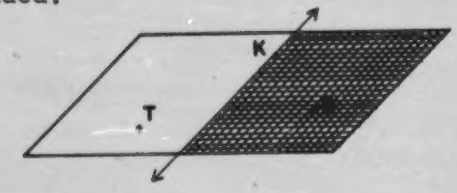


and three points, A, B, and C. Is $\overline{AB} \cap \ell$ the empty set? Is $\overline{BC} \cap \ell$ the empty set? What about $\overline{AC} \cap \ell$? We may say that:

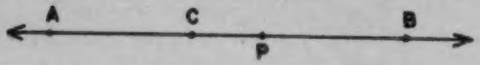
Any line ℓ of plane M separates the plane into two half-planes

We call the two half-planes into which ℓ separates M, the sides of ℓ. We indicate the sides of ℓ by referring to the A-side of ℓ or the C-side of ℓ. Notice that in the above figure, the B-side of ℓ = the A-side of ℓ . The line ℓ is not in either half-plane.

In the figure below the S-side of line k is shaded and the T-side of k is not shaded.



Now consider a line ℓ. How would you define a half-line? Can you say anything about segments in this definition as we did in defining half-planes and half-spaces? What would the boundary be? Is the boundary a set of points?



If point P separates the line in the figure into two half-lines, are A and B on the same half-line? Are A and C on the same half-line? Is P between B and C?

Our third case should now be clear. Can you state it?

It is important to note that these three cases are almost alike. They deal with the same idea in different situations. Thus:

(1) Any plane separates space into two *half-spaces*.

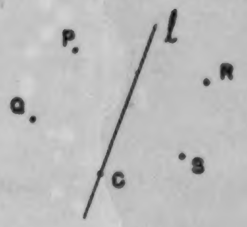
(2) Any line of a plane separates the plane into two *half-planes*.

(3) Any point of a line separates the line into two *half-lines*.

There is one other definition that is useful. A *ray* is a half-line together with its endpoint. A ray has one endpoint. A ray without its endpoint is a *half-line*. We usually draw a ray like this, ⟶. If A is the endpoint of a ray and B is another point of the ray, we denote the ray by $\overrightarrow{AB}$. Note that $\overrightarrow{BA}$ is not $\overrightarrow{AB}$. We use the term ray in the same sense in which it is used in "ray of light."

In everyday language, we sometimes do not distinguish between lines, rays, and segments. In geometry we should distinguish between them. A "line of sight" really refers to a ray. You do not describe somebody as in your line of sight if he is behind you. The right field foul line in baseball really refers to a segment and a ray. The segment extends from home plate through first base to the ball park fence. It stops at the fence. The ray starts on the ground and goes up the fence. What happens to a home run ball after it leaves the park makes no difference to the play in a major league game.

Exercises 4-7



1. In the figure at the right, is the R-side of ℓ the same as the S-side of ℓ ? Is it the same as the Q-side? Are the intersections of ℓ and $\overline{PQ}$, ℓ and $\overline{RS}$ empty? Are the intersections of ℓ and $\overline{QS}$, ℓ and $\overline{PR}$ empty? Considering the sides of ℓ, are the previous two answers what you would expect?
2. Draw a line containing points A and B. What is $\overrightarrow{AB} \cap \overrightarrow{BA}$? What is the set of points not in $\overrightarrow{AB}$?
3. Draw a horizontal line. Label four points of it A, B, C and D in that order from left to right.
   Name two rays (using these points for their description):
   a. Whose union is the line.
   b. Whose union is not the line, but contains A, B, C, and D.
   c. Whose union does not contain A.
   d. Whose intersection is a point.
   e. Whose intersection is empty.
4. Does a segment separate a plane? Does a line separate space?
5. Draw two horizontal lines k and ℓ on your paper. These lines are parallel. Label point P on ℓ. Is every point of ℓ on the P-side of k? Is this question the same as "Does the P-side of k contain ℓ "?
6. The idea of a plane separating space is related to the idea of the surface of a box separating the inside from the outside. If P is a point on the inside and Q a point on the outside of a box, does $\overline{PQ}$ intersect the surface?

*7. Explain how the union of two half-planes might be a plane.

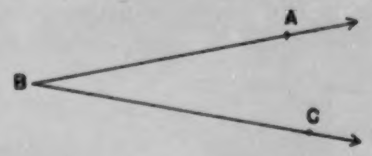
*8. If A and B are points on the same side of the plane M (in space), must $\overleftrightarrow{AB} \cap M$ be empty? Can $\overleftrightarrow{AB} \cap M$ be empty?

4-8. Angles and Triangles

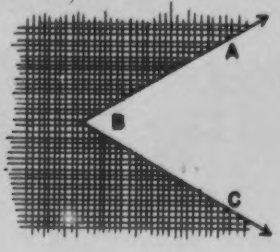
Angles

Some of the most important ideas of geometry deal with angles and triangles. An angle is a set of points consisting of two rays with an endpoint in common and not both on the same straight line. Let us say this in another way. Let $\overrightarrow{BA}$ and $\overrightarrow{BC}$ be two rays such that A, B, and C are not all on the same line. Then the set of points consisting of all the points of $\overrightarrow{BA}$ together with all the points of $\overrightarrow{BC}$ is called the angle ABC. An angle is the union of two rays. The point B is called the vertex of the angle. The rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are called the rays (or sometimes the sides) of the angle. An angle has exactly one vertex and exactly two rays.

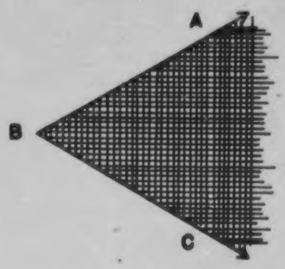
An angle is drawn in the figure below. You will recall from section 4-3 that we really mean "a representation of an angle is drawn." Three points of the angle are labeled so that the angle is read "angle ABC" and may be written as, "$\angle$ABC." The letter of the vertex is always listed in the middle. Therefore, $\angle$ABC is $\angle$CBA. Note that in labeling this angle the order of A and C does not matter, but B must be in the middle. Is $\angle$ABC the same as $\angle$BAC (not drawn)?



From the figure it looks as if the angle ABC separates the plane containing it. It is true that the angle does separate the plane. The two pieces into which the angle separates the plane look somewhat different. They look like:



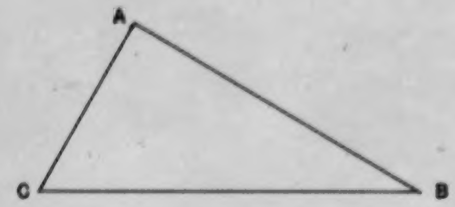
and



We call the piece on the right the interior of the angle and the one on the left the exterior. We can define the interior of the $\angle ABC$ as the intersection of the A-side of the line $\overleftrightarrow{BC}$ and the C-side of the line $\overleftrightarrow{AB}$. It is the intersection of two half planes and does not include the angle. The exterior is the set of all points of the plane not on the angle or in the interior.

Triangles

Let A, B, and C be three points not all on the same straight line. The triangle ABC, written as $\triangle ABC$, is the union of $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$. You will recall that the union of two sets consists of all of the elements of the one set together with all the elements of the other. We may define the $\triangle ABC$ in another way. The triangle ABC is the set of points consisting of A, B, and C, and all points of $\overleftrightarrow{AB}$ between A and B, all points of $\overleftrightarrow{AC}$ between A and C, and all points of $\overleftrightarrow{BC}$ between B and C. The points A, B, and C are the vertices of $\triangle ABC$. We say "vertices" when referring to more than one vertex. Triangle ABC is represented in the figure.



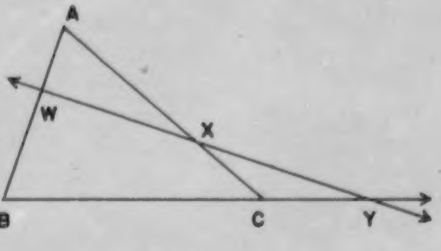
Angles of a Triangle

We speak of $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ as the sides of the triangle. We speak of $\angle ABC$, $\angle ACB$, and $\angle BAC$ as the angles of the triangle. These are the angles determined by the triangle. Are the sides of the triangle contained in the triangle? Are the angles of a triangle contained in the triangle? You may wonder why we call $\angle ABC$ an angle of $\triangle ABC$ when $\angle ABC$ is not contained in $\triangle ABC$. (Read again the paragraphs on Angles.) We speak of the graduates of a school even though graduates are not in the school.

Note that a triangle is a set of points in exactly one plane. Every point of the triangle ABC is in the plane ABC. Look at the figure on the previous page. Does ΔABC separate the plane in which it lies? Yes, it certainly seems to do so. It is true that it does. The ΔABC has an interior and an exterior. The interior is the intersection of the interiors of the three angles of the triangle. The exterior is the set of all points of plane ABC not on ΔABC or in the interior of ΔABC.

Exercises 4-8

1. Label three points A, B, and C not all on the same line. Draw $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{BC}$. (a) Shade the C-side of $\overleftrightarrow{AB}$. Shade the A-side of $\overleftrightarrow{BC}$. What set is now doubly shaded? (b) Shade the B-side of $\overleftrightarrow{AC}$. What set is now triply shaded?
2. Label three points X, Y, and Z not all on the same line.
   a. Draw $\angle XYZ$ and $\angle XZY$. Are they different angles? Why?
   b. Draw $\angle YXZ$. Is this angle different from both of the other two you have drawn?
   c. Each angle is a set of points in exactly one plane. Why is this true?
3. Draw a triangle ABC.
   a. In the triangle, what is $\overline{AB} \cap \overline{AC}$?
   b. Does the triangle contain any rays or half-lines?
   c. In the drawing extend $\overline{AB}$ in both directions to obtain $\overleftrightarrow{AB}$. What is $\overline{AB} \cap \overleftrightarrow{AB}$?
   d. What is $\overleftrightarrow{AB} \cap \Delta ABC$?



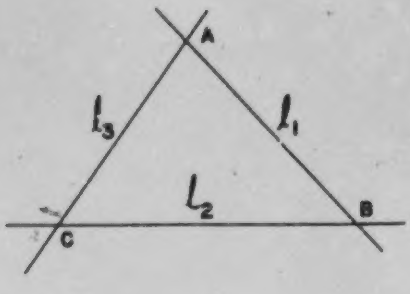
4. Refer to the figure on the right.
   a. What is $\overline{YW} \cap \Delta ABC$?
   b. Name the four triangles in the figure.
   c. Which of the labeled points, if any, are in the interior of any of the triangles?
   d. Which of the labeled points, if any, are in the exterior of any of the triangles?
   e. Name a point on the same side of $\overleftrightarrow{WY}$ as C and one on the opposite side.

5. Draw a figure like that of Exercise 4.
   a. Label a point P not in the interior of any of the triangles.
   b. Label a point Q inside two of the triangles.
   c. If possible, label a point R in the interior of $\triangle ABC$ but not in the interior of any other of the triangles.

6. If possible, make sketches in which the intersection of a line and a triangle is:
   a. The empty set
   b. A set of two elements
   c. A set of one element
   d. A set of exactly three elements

7. If possible, make sketches in which the intersection of two triangles is:
   a. The empty set
   b. Exactly two points
   c. Exactly four points
   d. Exactly five points.

8. In the figure, what are the following:
   a. $\angle ABC \cap \overleftrightarrow{AC}$
   b. $\triangle ABC \cap \overleftrightarrow{AB}$
   c. $\ell_1 \cap \angle ACB$
   d. $\overline{AB} \cap \ell_2$
   e. $\angle BCA \cap \angle ACB$
   f. $\overline{BC} \cap \angle ABC$
   g. $\overrightarrow{BC} \cap \angle ACB$
   h. $\angle ABC \cap \triangle ABC$



*9. In a plane if two triangles have a side of each in common, must their interiors intersect? If three triangles have a side of each in common, must some two of their interiors intersect?

*10. Draw $\angle ABC$. Label points X and Y in the interior and P and Q in the exterior.
   a. Must every point of $\overline{XY}$ be in the interior?
   b. Is every point of $\overline{PQ}$ in the exterior?
   c. Can you find points R and S in the exterior so that $\overline{RS} \cap \angle ABC$ is not empty?
   d. Can $\overline{XP} \cap \angle ABC$ be empty?

4-9. One-to-One Correspondences

In Chapter 3 we used the idea of "one-to-one correspondence" in talking about counting numbers. This idea is also useful in geometry. By "one-to-one correspondence" we mean the matching of each member of a certain set with a corresponding member of another set. Before we use this idea in geometry, let us review our previous experience.

One example used in Chapter 3 concerned primitive man as he kept track of his sheep by matching each sheep in his flock with a stone. As you remember, the shepherd would build a pile of stones each morning, composed of one stone for each sheep, as the flock left the fold for the pasture. In the evening he would transfer the stones to a new pile, one at a time, as each sheep entered the fold. If all stones were thus transferred, then the shepherd knew that all his sheep were safe in the fold for the night. This is true because for each stone there was a matching sheep, and for each sheep there was a matching stone.

Let us take another example. Suppose you have sold seventeen tickets to a school play. Let T be the set of tickets you have sold. Let C be the set of people who are admitted to the theatre by these tickets. Is there a one-to-one correspondence between T and C? How do you know?

Consider the set of counting numbers less than eleven. Let us form two sets from these numbers. Set A, containing the odd numbers: {1, 3, 5, 7, 9} and set B, containing the even numbers: {2, 4, 6, 8, 10}. Is there a one-to-one correspondence between set A and set B? The answer, of course, is yes, because every odd number can be matched with an even number. Let us demonstrate this by use of the following scheme:

$$\begin{array}{ccccc} 1 & 3 & 5 & 7 & 9 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 2 & 4 & 6 & 8 & 10 \end{array}$$

Is this the only way in which the elements of these two sets can be matched? Form a different matching of your own.

The game of musical chairs is fun because it is based on not having a one-to-one correspondence. Why is this? Is there ever a one-to-one correspondence at some stage in the game?

Given set A: {a, b, c, d, e, f, g}, and set B: {I, II, III, IV, V}. Is there a one-to-one correspondence between these two sets? Demonstrate your answer by using the scheme used previously for the sets of even and odd numbers.

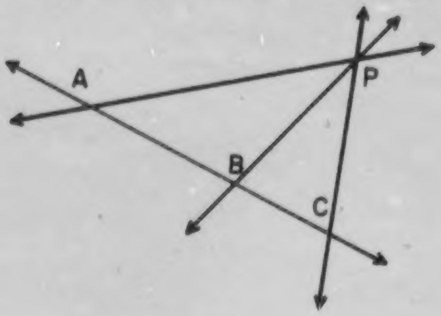
Exercises 4-9-a

1. Is there a one-to-one correspondence between the pupils and the desks in your room? Why?
2. Is there a one-to-one correspondence between the states of the United States and the U. S. cities of over 1,000,000 in population? Why?
3. You have heard of the expression: "Let's count noses." Does this imply a one-to-one correspondence situation? If so, what is it?
4. Show that there is a one-to-one correspondence between the set of even whole numbers and the set of odd whole numbers.
5. If set R is in one-to-one correspondence with set S and set S with set T, is set R in one-to-one correspondence with set T? Why?
6. BRAINBUSTER. Establish a one-to-one correspondence between the set of even whole numbers and the set of whole numbers.

Now let us see how we can use the idea of one-to-one correspondence in geometry.

Class Exercise and Discussion

Follow the given directions in making a drawing somewhat like the given figure.



Questions are scattered throughout the directions. Answer the questions as you go along.

1. Draw a line and label it ℓ.
2. Choose a point not on line ℓ and label it P.
3. Mark some point A on line ℓ.
4. Draw line $\overleftrightarrow{PA}$.
   a. Is $\overleftrightarrow{PA} \cap \ell$ equal to the empty set?
   b. Does the intersection set of PA and ℓ have only one element? Why?
5. Choose two other points, B and C on ℓ. Draw $\overleftrightarrow{PB}$ and $\overleftrightarrow{PC}$.
   a. Through each additional point marked on ℓ can you draw a line that also goes through point P?
   b. Let all lines which intersect ℓ and pass through P be the elements of a set called K. How many elements of K have been drawn up to now?
   c. Does each indicated element of K contain a point on ℓ?
   d. Can each indicated element of K be matched with an indicated point on ℓ?
   e. Do you think that, if more elements of K were drawn and more points on ℓ were marked, each element of K could be matched with a corresponding element of ℓ?

6. Mark points D and E on ℓ. Draw elements of K that can be matched with these points. Is it true that for each element (indicated or not) of K there is a corresponding element of ℓ and for each element of ℓ there is a corresponding element of K?
7. Draw an element of K.
   a. Does it intersect ℓ?
   b. In how many points does it intersect ℓ?
8. Copy and complete the following sentence to make it true: To a ________ through P and intersecting ℓ there corresponds a ________ on ℓ and to a ________ on ℓ there corresponds a ________ through P and intersecting ℓ. In other words, there is a one-to-one correspondence between the set K of lines and the set ℓ of points.

The facts you learn about number and the facts you learn in the study of geometry will eventually be combined in a subject known as "coordinate geometry." Great advances in science and technology have been made because of this combination. To understand how this union of number and geometry is accomplished, we shall once again call upon the ideas of "line in the geometric sense," "number line," and "one-to-one correspondence."

Exercises 4-9-b

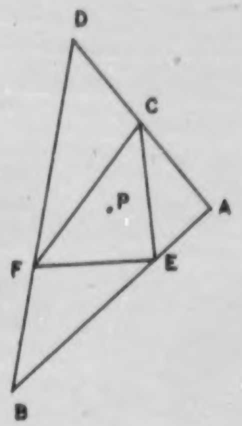
1. Describe a one-to-one correspondence between the points A, B, and C which determine a triangle and the sides of the triangle. Can you do this in more than one way?
2. Draw a triangle with vertices A, B, and C. Label a point P in the interior of $\triangle ABC$. Let H be the set of all rays having P an an endpoint. We understand that the elements of H are in the plane of $\triangle ABC$. Draw several rays of H. Can you observe a one-to-one correspondence between H and $\triangle ABC$? For every point of $\triangle ABC$ is there exactly one ray of H containing it? For every ray of H is there exactly one point of $\triangle ABC$ on such ray?

3. Draw an angle XYZ with the vertex at Y. Draw the segment $\overline{XZ}$. Think of K as a set of rays in plane XYZ with common endpoint at Y. K is the set of all such rays which do not contain points in the exterior of $\angle XYZ$. $\overrightarrow{YX}$ and $\overrightarrow{YZ}$ are two of the many elements of K. Draw another element of K. Does it intersect $\overline{XZ}$? For each element of K will there be one such matching point of $\overline{XZ}$? Label D a point of $\overrightarrow{YX}$ and E a point of $\overrightarrow{YZ}$. Draw $\overline{DE}$. Is there a similar one-to-one correspondence between the set of points of $\overline{XZ}$ and the set of points of $\overline{DE}$?

*4. Describe a one-to-one correspondence between the set of points on the surface of a ball and the set of rays with common endpoint inside the ball.

*5. Describe a one-to-one correspondence between a set H of all lines in a plane through a point and a set K of all planes through a line in space. (Think of a revolving door and the floor under the door.)

*6. Let S be the set of all rays in plane ABD with endpoint at P. (a) Is there a one-to-one correspondence between S and Δ ABD? (b) Is there a one-to-one correspondence between S and Δ FCE? (c) Is there a one-to-one correspondence between Δ ABD and Δ FCE? Why?



4-10. Simple Closed Curves

In newspapers and magazines you often see graphs like those in figures a and b. These graphs represent what are called curves. We shall consider curves to be special types of sets of points. Sometimes paths that wander around in space are thought of as curves. In this section, however, we confine our attention to curves that are contained in one plane. We can represent them by figures we draw on a chalkboard or on a sheet of paper.

A curve is a set of points which can be represented by a pencil drawing made without lifting the pencil off the paper. Segments and triangles are examples of curves we have already studied. Curves may or may not contain portions that are straight. In everyday language we use the term "curve" in this same sense. When a baseball pitcher throws a curve, the ball seems to go straight for a while and then "breaks" or "curves."

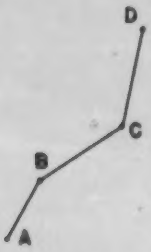


Fig. a

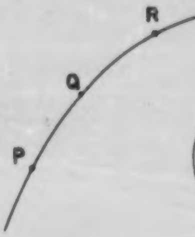


Fig. b

Fig. c

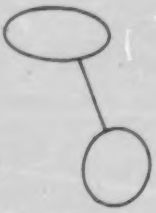


Fig. d

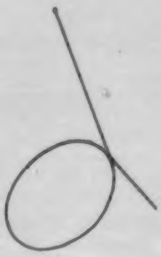


Fig. e

One important type of curve is called a broken-line curve. It is the union of several line segments; that is, it consists of all the points on several line segments. Fig. a represents a broken-line curve. A, B, C, and D are indicated as points on the curve. We also say that the curve contains or passes through these points. Figures b to i also represent curves. In Fig. b, points P, Q, and R are indicated on the curve. Of course, we think of the curve as containing many points other than P, Q, and R.

A curve is said to be a simple closed curve if it can be represented by a figure drawn in the following manner. The drawing starts and stops at the same point. Otherwise, no point is touched twice. Figures c, g, h, and i represent simple closed curves. The other figures of this section do not. Figure j represents two simple closed curves. The boundary of a state like Iowa or Utah on an ordinary map represents a simple closed curve. A fence which extends all the way around a cornfield suggests a simple closed curve.

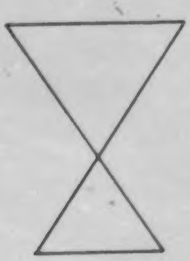


Fig. f

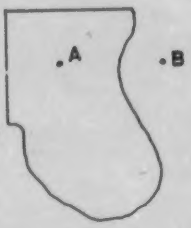


Fig. g

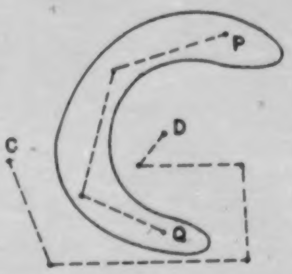


Fig. h

The examples we have mentioned, including that of a triangle, suggest a very important property of simple closed curves. Each simple closed curve has an interior and an exterior in the plane. Furthermore, any curve at all containing a point in the interior and a point in the exterior must intersect the simple closed curve. As an example, consider any curve containing A and B of Fig. g and lying in the plane. Also any two points in the interior (or any two points in the exterior) may be joined by a broken-line curve which does not intersect the simple closed curve. Fig. h indicates this. A simple closed curve is the boundary of its interior and also of its exterior.

We call the interior of a simple closed curve a region. There are other types of sets in the plane which are also regions. In Fig. j, the portion of the plane between the two simple closed curves is called a region. Usually a region (as a set of points) does not include its boundary.

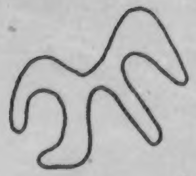


Fig. i

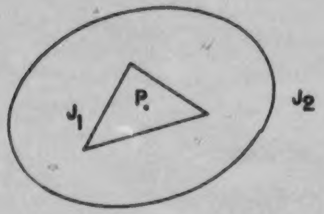


Fig. j

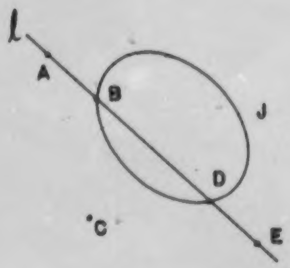
Consider Figure j. The simple closed curve (represented by) J_1 is in the interior of the simple closed curve J_2. We may obtain a natural one-to-one correspondence between J_1 and J_2 as follows. Consider a point such as P in the interior of J_1. Consider the set of rays with endpoint at P. Each such ray intersects each of J_1 and J_2 in a single point. Furthermore, each point of J_1 and each point of J_2 is on one such ray. A point of J_1 corresponds to a point of J_2 if the two points are on the same ray from P. Note that this procedure using rays would not determine a one-to-one correspondence if one of the simple closed curves were like that in Fig. i.

Exercises 4-10

1. Draw a figure representing two simple closed curves whose intersection is exactly two points. How many simple closed curves are represented in your figure?
2. In Fig. j, describe the region between the curves in terms of intersection, interior and exterior.
3. Draw two triangles whose intersection is a side of each. Is the union of the other sides of both triangles a simple closed curve? How many simple closed curves are represented in your figure?
4. In a map of the United States, does the union of the boundaries of Colorado and Arizona represent a simple closed curve?

5. Think of X and Y as bugs which can crawl anywhere in a plane. List three different simple sets of points in the plane, any one of which will provide a boundary which separates X and Y.

6. The line ℓ and the simple closed curve J are as shown in the figure.



   a. What is $J \cap \ell$?
   b. Draw a figure and shade the intersection of the interior of J and the C-side of ℓ.
   c. Describe in terms of rays the set of points on ℓ not in the interior of J.

*7. Draw two simple closed curves, one in the interior of the other such that, for no point P do the rays from P establish a one-to-one correspondence between the two curves. Consider Fig. 1.

*8. Draw two simple closed curves whose interiors intersect in three different regions.

9. BRAINBUSTER. Explain why the intersection of a simple closed curve and a line cannot contain exactly three points if the curve crosses the line when it intersects it.

CHAPTER 5

FACTORING AND PRIMES

5-1. Primes

In previous chapters we studied some of the properties of the counting numbers. Here we shall discuss how they can be expressed as products of other counting numbers. For instance:

$$6 = 2 \times 3 = 1 \times 2 \times 3 = 1 \times 3 \times 2 = 1 \times 6 = 6 \times 1.$$

$$5 = 1 \times 5 = 5 \times 1 = 5 \times 1 \times 1.$$

$$12 = 2 \times 2 \times 3 = 4 \times 3 = 1 \times 2 \times 6 = 1 \times 3 \times 4.$$

Are there other ways in which these numbers can be expressed as products of counting numbers? Express the following as products of counting numbers in various ways: 15, 18, 30.

In the products listed above which are equal to 6, we see that 1, 2, 3 and 6 divide exactly into 6. That is, if 6 is divided by any one of these four numbers, the remainder is zero. Similarly 1 and 5 are the only counting numbers that divide exactly into 5; while 1, 2, 3, 4, 6 and 12 are those which divide exactly into 12. Two other ways of making the same statement are:

1) The number 6 is divisible by 1, 2, 3 and 6.
2) The number 6 is a multiple of 1, 2, 3 and 6.

Thus 5 is divisible by 1 and 5, or 5 is a multiple of 1 and 5; also, 12 is divisible by 1, 2, 3, 4, 6 and 12, or 12 is a multiple of each of the numbers 1, 2, 3, 4, 6 and 12.

On the other hand, 12 is not divisible by 5 since if 12 is divided by 5 the remainder is 2. For a similar reason, 6 is not divisible by 4.

From the point of view of this section, the number 1 is in a class by itself since every counting number is a multiple of 1; that is, every counting number is divisible by 1. It is not true that every counting number is divisible by 2 (3 is not); not every counting number is divisible by 23 (24 is not); not every counting number is divisible by 1976 (5 is not).

Every counting number is a multiple of 1 as we have seen. What are the multiples of 2 which are greater than 2? Let us look at one way to answer this question systematically: First write down the numbers, for instance, from 1 to 30 inclusive. The first multiple of 2 greater than 2 is 4; cross out the 4 and every second number after that. To keep track, write a 2 below each number you have crossed out. The list will then look like the following:

$1 \quad 2 \quad 3 \quad \cancel{4}_2 \quad 5 \quad \cancel{6}_2 \quad 7 \quad \cancel{8}_2 \quad 9 \quad \cancel{10}_2 \quad 11 \quad \cancel{12}_2$
$13 \quad \cancel{14}_2 \quad 15 \quad \cancel{16}_2 \quad 17 \quad \cancel{18}_2 \quad 19 \quad \cancel{20}_2 \quad 21 \quad \cancel{22}_2 \quad 23 \quad \cancel{24}_2$
$25 \quad \cancel{26}_2 \quad 27 \quad \cancel{28}_2 \quad 29 \quad \cancel{30}_2$

We neither cross out 2 nor write a 2 under it because that is the number whose multiples we are considering. The numbers above which are not crossed out are 1, 2, and the numbers less than 31 which are not multiples of 2.

Our second step would be to go through the same table and cross out the multiples of 3 which are greater than 3. Then the table would look like this:

$1 \quad 2 \quad 3 \quad \cancel{4}_2 \quad 5 \quad \cancel{6}_{2,3} \quad 7 \quad \cancel{8}_2 \quad \cancel{9}_3 \quad \cancel{10}_2 \quad 11 \quad \cancel{12}_{2,3} \quad 13$
$\cancel{14}_2 \quad \cancel{15}_3 \quad \cancel{16}_2 \quad 17 \quad \cancel{18}_{2,3} \quad 19 \quad \cancel{20}_2 \quad \cancel{21}_3 \quad \cancel{22}_2 \quad 23$
$\cancel{24}_{2,3} \quad 25 \quad \cancel{26}_2 \quad \cancel{27}_3 \quad \cancel{28}_2 \quad 29 \quad \cancel{30}_{2,3}$

Here we have crossed out every third number beginning with 6, but we have not crossed out 3 since that is the number whose multiples we are finding. (Some of the multiples of 3 had already been crossed out since they were also multiples of 2.) Except for the numbers 1, 2, and 3, none of the numbers are multiples of either 2 or 3.

As a class exercise, write out the numbers from 1 to 100 inclusive. First, cross out all multiples of 2 and 3 except 2 and 3 as we did above. The number 4 and all multiples of 4 are already crossed out since any multiple of 4 is also a multiple of 2. The next number not crossed out is 5. So for the third step cross out every fifth number after 5 (that is, beginning with 10), and write a 5 below each number crossed out. For the fourth and fifth steps, similarly cross out multiples of 7 and 11 which are greater than 7 and 11. Keep track of the multiples as indicated. Did you cross out any new numbers when you were considering multiples of 11? Would we cross out any new numbers if we considered multiples of 12? of 13?

From the way in which the table was constructed you see that every number crossed out is a multiple of a smaller number different from 1. These numbers are called composite numbers.

Definition: A composite number is a counting number which is divisible by a smaller counting number different from 1.

The table which you have constructed with numbers crossed out as directed is called the "Sieve of Eratosthenes" for the first 100 numbers. It is called a "sieve" because in it you have sifted out all the composite numbers less than 100. Notice that when we crossed out the multiples of 2 and 3 less than 31, the composite number 25 remained. However, the number 25 was eliminated when we crossed out multiples of 5 in the third step. Similarly, the number 49 was not crossed out in the Sieve of Eratosthenes until we crossed out multiples of 7.

Except for the number 1, the numbers of the Sieve of Eratosthenes which are not crossed out are called prime numbers.

Definition: A prime number is a counting number, other than 1, which is divisible only by itself and 1.

Since it eliminates the composite numbers, the Sieve of Eratosthenes is a good way of finding a list of all prime numbers up to a certain point. The composite numbers are sifted out. The prime numbers remain. Why are the remaining numbers prime numbers?

The number 1 is not included in the set of primes partly because it is divisible by itself only. We shall have another stronger reason for this later on.

Exercises 5-1

1. a. List the prime numbers less than 100. b. Now list the prime numbers less than 130 but greater than 100.
2. a. How many prime numbers are less than 50?
   b. How many prime numbers are less than 100?
   c. How many prime numbers are less than 130?

Do problems 3, 4, and 5 first without using Eratosthenes' Sieve and then use it to check your results:

3. List all the multiples of 5 which are less than 61.
4. List the set of numbers less than 50 which are multiples of 7.
5. List the set of numbers which are less than 100 and are also multiples of both 3 and 5.
6. In the table below, the numbers along the top represent values of a and those down the left side represent values of b. In each case if a is divisible by b, write the values of $\frac{a}{b}$ in the a-column and b-row. If a is not divisible by b, write "no" in the a-column and b-row.

	a=12	14	17	18	20	25	27
b=1							
b=2							
b=3							
b=4							
b=5							
b=6							
b=7							

7. Express each of the following counting numbers as a product of two smaller counting numbers or indicate that it is impossible to do this:
   a. 12
   b. 36
   c. 31
   d. 7
   e. 8
   f. 11
   g. 35
   h. 5
   i. 39
   j. 42
   k. 6
   l. 41
   m. 82
   n. 95
8. a. By what numbers is 24 divisible?

   b. The number 24 is a multiple of what numbers?
   c. Are the two sets of numbers you have found in a and b the same? Why or why not?

9. Write 12 in all possible ways as a product of counting numbers greater than 1.
10. List the pairs of prime numbers less than 100 which have a difference of 2. How many are these? Such pairs are called twin primes.
11. Express each even number between 4 and 22 as a sum of two prime numbers. (An even number, recall, is one divisible by 2). Most mathematicians believe that every even number greater than 2 is the sum of two prime numbers but no one has been able to prove it.
12. Are there three numbers that might be called prime triplets?
13. a. Locate the numbers from 1 to 50 along a number line.
    b. Underline the numerals in every second position, starting with 1.
    c. Circle the numerals for the prime numbers.
    d. Did you need to circle any numeral that was not underlined? If so, write all such numerals.
14. What is the intersection of the set of prime numbers and the set of odd numbers less than 30?

5-2. Factors

The word "factor" is commonly used in mathematics. Though the term may be new to you, the idea is not. We know that $5 \times 2 = 10$. Instead of calling one of the numbers the multiplicand and the other one the multiplier, we give both of them the same name -- factor. Thus, 5 and 2 are factors of 10; 6 and 7 are factors of 42, since $6 \times 7 = 42$. Also, $42 = 2 \times 3 \times 7$; so 2, 3, and 7 are factors of 42.

Example 1: Write 12 as a product of factors.

$12 = 1 \times 12,$

or $12 = 2 \times 6,$

or $12 = 3 \times 4,$

or $12 = 2 \times 2 \times 3 = 2^2 \times 3$

When we say "the factors" we mean "all the factors" of a number. For example, the number 6 has four factors, 1, 2, 3, and 6. The number one and the number itself are always factors of a number.

Example 2: Find the set of factors of 20.

The set of factors of 20 is $\{1, 2, 4, 5, 10, 20\}$.

The idea of factors is associated with multiplication. In mathematical symbols we define factor the following way:

Definition. If a, b, and c are whole numbers and if $ac = b$, then the number a is called a factor of b. (Under these conditions c is also a factor of b.)

Using the terms of the first section, we say that 3 is a factor of 12 because 12 is divisible by 3. In the symbols of the definition, we see that the number a is a factor of b if b is divisible by a.

The number 1 has only one factor, itself. Each prime number has exactly two factors, itself and 1. A composite number has how many factors?

Consider the number 24. It can be written as 4×6. Both 4 and 6 are composite numbers since they can be written as products of smaller counting numbers: $4 = 2 \times 2$ and $6 = 2 \times 3$. Thus

$$24 = 2 \times 2 \times 2 \times 3.$$

However, 2 and 3 are prime numbers since they cannot be expressed as products of smaller numbers. We cannot go any further in this process. We therefore say that $2 \times 2 \times 2 \times 3$ is a complete factorization of 24.

Definition: If a counting number is written as a product of prime numbers, this product is called a complete factorization of the given number.

Example 1: Find a complete factorization of 20.

$$20 = 4 \times 5 = 2 \times 2 \times 5 = 2^2 \times 5.$$

Here 4×5 is not a complete factorization of 20 since 4 is not a prime number, but $2 \times 2 \times 5$ and $2^2 \times 5$ are complete factorizations. The most compact complete factorization of 20 is $2^2 \times 5$.

Example 2: Find a complete factorization of 72.

Method I

$72 = 8 \times 9$
$72 = (4 \times 2) \times (3 \times 3)$
$72 = (2 \times 2) \times 2 \times (3 \times 3)$
$72 = (2 \times 2 \times 2) \times (3 \times 3)$

Using exponents,

$72 = 2^3 \times 3^2$

Method II

Using continuing short division

$$\begin{array}{r|l} 2 & 72 \\ \hline 2 & 36 \\ \hline 2 & 18 \\ \hline 3 & 9 \\ \hline & 3 \\ \hline \end{array}$$

$72 = 2^3 \times 3^2$

We might have used a fewer number of steps.

Notice that in both examples, the only factors appearing in the last products are prime numbers. Not all the factors of 20 and 72 (such as 4) appear in the final complete factorization. It is convenient but not necessary to use exponents wherever possible.

Note that $2 \times 5 \times 2$ is also a complete factorization of 20, but this is the same as $2 \times 2 \times 5$ except for the order of the factors. Similarly, in the factorization of 72, $2^2 \times 3 \times 2$ is the same as $2^3 \times 3$ except for the order of the factors. In fact, a very fundamental property of the counting numbers is that there is only one way to write a complete factorization of any counting number except for the order in which the prime factors appear. This property is given a special name:

The Unique Factorization Property of the Counting Numbers:

Every counting number greater than 1 can be factored into primes in only one way except for the order in which they occur in the product.

The word "unique" may be unfamiliar to some of you in this sense. It merely means that there is *only one* factorization except for order. One might say that the Empire State Building is unique because there is no other building like it.

Here we have another reason for excluding 1 from the set of prime numbers. If we had called 1 a prime, then 5 could have been expressed as a product of primes in many different ways: 5×1, $5 \times 1 \times 1$, $5 \times 1 \times 1 \times 1$, ... Here the product would not be unique except for the order in which the factors are written.

Exercises 5-2

1. List the set of factors for each of the following:
   a. 10 b. 15 c. 9 d. 13 e. 27 f. 24 g. 11
2. Factor the numbers listed in as many ways as possible using only two factors each time. Because of the commutative property, we shall say that $3 \cdot 5$ is not different from $5 \cdot 3$.
   a. 10
   b. 15
   c. 9
   d. 100
   e. 24
   f. 16
   g. 72
   h. 81
3. Write a complete factorization of:
   a. 10 b. 15 c. 9 d. 30 e. 45 f. 50 g. 13
4. According to our definition of factor, is zero a factor of 6? Is 6 a factor of zero? Explain your answers.
5. a. What factors of 20 do not appear in a complete factorization of 20? b. What factors of 72 do not appear in a complete factorization of 72?
6. Find a complete factorization of:
   a. 105
   b. 42
   c. 75
   d. 300
   e. 64
   f. 345
   g. 311
   h. 1000
   i. 301
   j. 323

Definition. If a whole number is divisible by two it is an *even number*. If a whole number is not divisible by two it is an *odd number*.

7. Tell whether these numbers are odd or even:
   - a. 2×5
   - b. $3 + 7$
   - c. $6 \times 5 \times 3$
   - d. $2 + 16$
   - e. $7 + 8$
   - f. $3 \times 2 \times 6$
   - g. $128 - 37$
   - h. $3 \times 3 \times 7$
   - i. $3 \cdot (4 + 7)$
   - j. $5 \cdot (9 + 13)$

8. Classify each of the following as odd or even:
   - a. 11_{three}
   - b. 12_{five}
   - c. 33_{five}
   - d. 101_{two}

9. From the results of **Problem** 8 would you say that divisibility is a property of a numeral or a property of a number? Explain your answer.

*10. Copy the following table for counting number N and complete it through N = 30.

N	Factors of N	Number of Factors	Sum of Factors
1	1	1	1
2	1,2	2	3
3	1,3	2	4
4	1,2,4	3	7
5	1,5	2	6
6	1,2,3,6	4	12
7	1,7	2	8
8	1,2,4,8	4	15

   - a. Which numbers represented by N in the table above have exactly two factors?
   - b. Which numbers N have exactly three factors?
   - c. If $N = p^2$ (where p is a prime number), how many factors does N have?
   - d. If $N = pg$ (where p and g are different prime numbers), how many factors does N have? What is the sum of its factors?
   - e. If $N = 2^k$ (where k is a counting number), how many factors does N have?

f. If $N = 3^k$ (where k is a counting number), how many factors does N have?

g. If $N = p^k$ (where p is a prime number and k is a counting number), how many factors does N have?

h. Which numbers have $2N$ for the sum of their factors? These numbers are called *perfect numbers*. It is unknown how many perfect numbers there are or whether there are any odd perfect numbers.

5-3. Divisibility

To find the factors of a number, we can always guess and try, but it is much easier if we can tell from looking at a number whether or not it has a given factor. From Chapter 2 or from Sieve of Eratosthenes it is clear that a number written in the decimal system is even if the last digit is even. At least this is true as far as the sieve we have constructed goes. Thus:

A counting number written in the decimal system is even if its last digit is one of 0, 2, 4, 6, 8. If its last digit is not one of these, it is odd.

Suppose we see why this is so. To do this, remember how we found the multiples of 2 when we began to construct the Sieve of Eratosthenes. We started with the number and added 2 again and again. The last digits repeated in the pattern: 2,4,6,8,0, 2,4,6,8,0,... This would continue no matter how far we extended the table. This shows that the even numbers are those whose last digit is one of the five numbers: 2, 4, 6, 8, 0.

In Problem 4 below you are asked to start with 5 and add 5 again and again to show the following:

A counting number expressed in the decimal system is divisible by 5 if its last digit is 0 or 5. Otherwise it is not divisible by 5.

What about divisibility by 3? Can we tell by looking at the last digit? The first ten multiples of 3 are

0, 3, 6, 9, 12, 15, 18, 21, 24, 27.

Each of the possible last digits, 0,1,2,3,4,5,6,7,8, and 9, appears in this list. On the other hand, none of the following are divisible by 3 even though each of the possible last digits appears here also:

4, 7, 10, 13, 16, 19, 22, 25, 28, 31.

We can see, then, that we cannot tell whether a number is divisible by 3 by looking at the last digit.

But suppose we add the digits of the multiples of 3. For 12 we have 1 + 2 = 3; for 15 we have 1 + 5 = 6; for 18 we have 1 + 8 = 9. By this means we can form the following table:

Multiple of 3	0	3	6	9	12	15	18	21	24	27	30	33	36	39
Sum of digits	0	3	6	9	3	6	9	3	6	9	3	6	9	12

Multiple of 3	42	45	48	51	54	57	60	63	66	69	72
Sum of Digits	6	9	12	6	9	12	6	9	12	15	9

Can you make any statement that seems to be true about the sum of the digits for all multiples of 3? You will see that in each case the sum of the digits is divisible by 3. Furthermore, if you add the digits of any number that is not divisible by 3 (take 25 where the sum of the digits is 7), the sum of the digits is not divisible by 3. Can you see why this will be true for all numbers? See Problem 3 in the next set.

You may notice that every third sum of digits in the table above is divisible by 9 and every third multiple of 3 is divisible by 9. Hence we have the following test for divisibility by 9.

A number is divisible by 9 if the sum of its digits is divisible by 9. Otherwise it is not divisible by 9.

There are also somewhat more complicated tests for divisibility by 11 and 13. These are dealt with in a supplementary unit on divisibility.

Suppose we apply what we have learned about divisibility to a few examples:

Example 1. Find a complete factorization of 232. Since the given number has 2 as its last digit, it is even and has 2 as a factor. So $232 = 2 \times 116$. Then 116 has 2 as a factor and we have $232 = 2^2 \times 58$. Then we have $232 = 2^3 \times 29$. We can see that 29 is a prime number by looking at our table of the Sieve of Eratosthenes or by trying the prime factors: 2,3,5,7,11,13,17,19,23 less than 29. Some of you may be able to see why it is necessary only to try 2,3, and 5.

A tabular way of finding the complete factorization is the following:

232	116	58	29
2	2	2	29

where 2 is the first factor and 116 is the quotient; then 2 is a factor of 116 and 58 is the quotient, etc. A complete factorization then is on the second line.

Example 2. Find a complete factorization of 573. Here the last digit is odd and hence 2 is not a factor. But the sum of the digits is 15 which is divisible by 3. Hence 3 is a factor of 573 and, dividing, we have $573 = 3 \times 191$. By our tests 2, 3, and 5 are not factors of 191. Trial shows that 7, 11, and 13 are not factors and hence 191 is a prime number. Why is it not necessary to try any primes larger than 13? Therefore, $573 = 3 \times 191$ is the complete factorization.

Example 3. Find a complete factorization of 539. Our tests show that none of 2,3,5 are factors. If we try 7 we see that $539 = 7 \times 77 = 7^2 \times 11$ which is a complete factorization.

It is important to notice that the tests for divisibility which have been given in this section depend on the number being written in the decimal system. For instance, the number 21 in the decimal system is written 30_{seven} in the system base seven. This number 30_{seven} is not even in spite of the fact that its last digit is zero. However, since 30_{seven} means $(3 \times \text{seven}) + 0$, the fact that the last digit is zero tells us that the number is divisible by seven. If a number is written to the base seven it is very easy to tell whether or not it is divisible by seven; one merely looks to see if the last digit is zero.

The property of one number being a factor of another does not depend on the way it is written; for instance, seven is always a factor of twenty-one, no matter how it is written. But the tests for divisibility which we have given here depend on the system of numeration in which the number is written.

Exercises 5-3

1. Find the smallest prime factor of each of the following:
   a. 115 b. 135 c. 321 d. 484 e. 539 f. 121
2. Find a complete factorization of each of the following:
   a. 39 c. 81 e. 180 g. 378 i. 576 k. 1098
   b. 60 d. 98 f. 258 h. 432 j. 729 l. 2324
3. Notice the list of multiples of 3. In going from 9 to 12, the units digit decreases from 9 to 2 and the tens digit increases from 0 to 1; hence the sum of the digits decreases by 7 - 1, or a net decrease of 6. Similarly in going from 18 to 21, the first digit increases by 1 and the second decreases by 7. Is this always true when the tens digit increases by 1? What happens when one goes from 99 to 102, from 999 to 1002, etc? Can you see from this, that always for a multiple of 3, it is true that the sum of its digits is a multiple of 3?
4. Show that the test given for divisibility by 5 always works.
5. List the multiples of 9 and see if you can show from this the test for divisibility by 9.
6. Can you give a test for divisibility of 6 in the decimal system?
7. Can you give a test for divisibility by 15 in the decimal system?
8. Which of the following numbers are divisible by 2:
   a. 1111_{ten} b. 1111_{seven} c. 1111_{six} d. 1111_{three}

9. Suppose a number is written in the system to the base seven. Is it divisible by ten if its last digit is zero? Is it divisible by three if the sum of its digits is divisible by three?

*10. Answer the above questions for a system of numeration to the base twelve.

*11. Find a test for divisibility by 6 in a system of numeration to the base seven.

*12. Give a test for divisibility by 4 in the decimal system.

5-4. Greatest Common Factor

Consider the numbers 10 and 12. We see that both 10 and 12 are even numbers. They are both divisible by 2, or we may say that 10 and 12 are multiples of 2. Because 2 is a factor of 10 and is also a factor of 12, we say that 2 is a "common factor" of 10 and 12.

All whole numbers are multiples of 1. Thus 1 is a common factor of the members of any set of whole numbers. Therefore, when we are looking for common factors we generally look for numbers other than 1.

What factor other than 1 is common to both 12 and 15? Is 2 a common factor? Since 15 is odd, 2 is not a factor of 15. Therefore it is impossible for 2 to be a common factor of 12 and 15. However, 12 and 15 are both multiples of 3. Hence, 3 is a common factor of 12 and 15.

Do the numbers 12 and 30 have any common factors? Writing the set of factors of 12 and the set of factors of 30 as shown at the right we see that there are several common factors. The numbers 1, 2, 3, and 6 are the common factors of 12 and 30.

Set of factors of 12 is {1,2,3,4,6,12}

Set of factors of 30 is {1,2,3,5,6,10,15,30}

Do the numbers 10 and 21 have any common factors?

Writing the set of factors of 10 and the set of factors of 21 as shown at the right we see that 10 and 21 do not have any common factors other than 1.

Set of factors of 10 is {1,2,5,10}

Set of factors of 21 is {1,3,7,21}

So we see that for any set of whole numbers the numbers have the common factor 1. For some sets of whole numbers there is a common factor other than 1, and, for some sets of whole numbers there are several common factors other than 1.

Recognizing common factors is useful in many ways. You have already used the idea of common factor in changing fractions to lower terms. For example, in changing $\frac{10}{12}$ to $\frac{5}{6}$ you use the common factor 2 of 10 and 12.

For $\frac{12}{30}$ we should see that 2 is a common factor of 12 and 30. The result is $\frac{6}{15}$. However, we see that for $\frac{6}{15}$ there is a common factor 3 of 6 and 15. Thus $\frac{6}{15}$ may be written as $\frac{2}{5}$.

Is it possible to change $\frac{12}{30}$ to $\frac{2}{5}$ using a single number instead of using 2 and 3 in turn. Some of you may have wondered why anyone would choose to change $\frac{12}{30}$ by using both 2 and 3 when it would be much quicker to use 6.

Is 6 a factor of both 12 and 30? Referring to the earlier listing of these factors, we see that 12 and 30 have the common factors 1, 2, 3, and 6. How does 6 differ from the other common factors? It is the largest of the common factors of 12 and 30. Such a factor is called the "greatest common factor."

Definition: The greatest common factor of two whole numbers is the largest whole number which is a factor of each of them.

Generally, the greatest common factor is more useful in mathematics than other common factors. Therefore, we are most interested in the greatest common factor.

Let's try another example. Suppose we wish to find the greatest common factor of 12 and 18. We could write the set of factors of each:

Set of factors of 12 is {1, 2, 3, 4, 6, 12}
Set of factors of 18 is {1, 2, 3, 6, 9, 18}

The set of common factors of 12 and 18 is {1, 2, 3, 6}. The largest member of the set is 6. Therefore, 6 is the greatest common factor of 12 and 18.

Similarly, suppose we wish to find the greatest common factor of 24 and 60. Writing the factors of each:

Set of factors of 24 is {1, 2, 3, 4, 6, 8, 12, 24}
Set of factors of 60 is {1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60}

The set of common factors is {1, 2, 3, 4, 6, 12}. The greatest of these factors is 12. Therefore, 12 is the greatest common factor of 24 and 60.

Exercises 5-4

1. Write the set of all factors for each of the following. List these carefully as you will use these sets in answering Problem 2 below.
   a. 6
   b. 8
   c. 12
   d. 15
   e. 16
   f. 21
2. Using your answers in Problem 1 above, write the set of common factors in each of the following cases:
   a. 6,8
   b. 8,12
   c. 12, 15
   d. 6,8,12
   e. 12,15,21
   f. 8,12,16
3. Write the set of all factors for each of the following.
   a. 19
   b. 28
   c. 36
   d. 40
   e. 45
   f. 72
4. Using your answers to Problems 1 and 3 above, write the set of common factors for each of the following.
   a. 19,28
   b. 16,36
   c. 28,40
   d. 36,45
   e. 40,72
   f. 19,36,45

5. Using your answers to Problems 2 and 4 above, write the greatest common factor for each of the following cases:

   a. 8,12,16
   c. 28,40
   e. 40,72
   b. 16,36
   d. 36,45
   f. 8,12,16,36

6. Find the greatest common factor in each of the following cases:

   a. 15, 25
   b. 18, 30
   c. 24, 36
   d. 25, 75
   e. 32, 48
   f. 15, 30, 36
   g. 12, 24, 48
   h. 40, 48, 72
   i. 15, 30, 45
   j. 20, 50, 100

7. a. What is the greatest common factor of 6 and 6?
   b. What is the greatest common factor of 29 and 29?
   c. What is the greatest common factor of a and a where a is any counting number?

8. a. What is the greatest common factor of 1 and 6?
   b. What is the greatest common factor of 1 and 29?
   c. What is the greatest common factor of 1 and a where a represents any whole number?

9. Let a and b represent any two different whole numbers where $a < b$.
   a. Will a and b always have a common factor? If so, what is the factor?
   b. Let c represent a common factor of a and b. Can $c = a$? Is so, give an example.
   c. Can $c = b$? If so, give an example.

10. Suppose 1 is the greatest common factor of three numbers.
    a. Must one of the three numbers be a prime number? If not, write a set of three composite numbers whose greatest common factor is 1.
    b. Can two of the numbers have a greatest common factor larger than 1. If so, give an example.

11. Let A be the set of all factors of 18. Let B be the set of all factors of 42.
    a. Write the set of factors of A.
    b. Write the set of factors of B.
    c. What is the intersection set of set A and set B?
    d. What are the common factors of 18 and 42?
    e. How do the answers for parts c and d compare?
12. If C is the set of factors of 30 and D is the set of factors of 51, what is the intersection set of sets C and D?
13. If E is the set of factors of 39 and G is the set of factors of 52, what is the intersection set of E and G?
14. In finding the greatest common factor for a set of numbers it is sometimes troublesome to write out all the factors. Try to find a shorter way of obtaining the greatest common factor. Assume that you are to find the greatest common factor of 36 and 45.
    a. Write a complete factorization of 36 and of 45. (List all of the prime factors of 36 and of 45).
       Example: $36 = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2$
       $45 = ? \cdot ? \cdot ? \quad = ? \cdot ?$
    b. What is the greatest common factor of 36 and 45?
    c. Compare the list of prime factors of 36 and 45 and the greatest common factor of 36 and 45. Can you see a shorter way of obtaining the greatest common factor?
15. a. Write a complete factorization for 18 and for 90.
    b. What is the greatest common factor of 18 and 90?
16. Factor completely each number in the following sets and find the greatest common factor for each set of numbers.
    a. {24, 60}
    b. {36, 90}
    c. {72, 108}
    d. {25, 75, 125}
    e. {24, 60, 84}
    *f. {42, 105, 147}
    *g. {165,234}
    *h. {306, 1173}
    *i. {2040, 2184}

*17. a. What is the greatest common factor of 0 and 6?

b. What is the smallest common factor of 0 and 6?

c. What is the smallest common factor for any two whole numbers?

*18. You have learned about operations with whole numbers; addition, subtraction, multiplication, and division. In this section we studied the operation of finding the greatest common factor. This is sometimes abbreviated G.C.F. For this problem only let us use the symbol " Δ " for the operation G.C.F. For any whole numbers, a and b and c,

$a \,\Delta\, b$ = G.C.F. for a and b

or $a \,\Delta\, c$ = G.C.F. for a and c

Example $12 \,\Delta\, 18 = 6$

$9 \,\Delta\, 15 = 3$

a. Is the set of whole numbers closed under the operation Δ ?

b. Is the operation Δ commutative; that is, does $a \,\Delta\, b = b \,\Delta\, a$?

c. Is the operation Δ associative, that is, does $a \,\Delta\, (b \,\Delta\, c) = (a \,\Delta\, b) \,\Delta\, c$?

5-5. Remainders in Division

Another way to find the greatest common factor is to make use of a relationship among the parts of a division problem. To understand this method let us review the division process.

The question "What is the result of dividing 16 by 5?" may be stated "How many 5's are contained in 16?" We can find the answer by repeated subtraction as shown at the right. By counting the number of times a 5 is subtracted we obtain the answer 3 with a remainder 1.

$$\begin{array}{r} 16 \\ -\ 5 \\ \hline 11 \\ -\ 5 \\ \hline 6 \\ -\ 5 \\ \hline 1 \end{array}$$

Does $16 = (5 + 5 + 5) + 1$?

The usual way of finding the answer to this division problem is shown below:

$$\begin{array}{rl} & 3 \text{ Remainder } 1 \\ 5 & \overline{)16} \\ & \underline{15} \\ & 1 \end{array}$$

To check the answer we use the following idea:

$$16 = (5 \times 3) + 1.$$

In the division problem above, the 16 is called the dividend, the 5 is the divisor, the 3 is the quotient, and the 1 is the remainder.

Let's try another example. Divide 253 by 25.

$$\begin{array}{rr} & 10 \\ 25 & \overline{)253} \\ & \underline{25} \\ & 3 \end{array}$$

Does $253 = (25 \times 10) + 3$?

In general, for any division problem:

$$\text{dividend} = (\text{divisor} \times \text{quotient}) + \text{remainder}$$

Using mathematical symbols, where

"a" represents the dividend,
"b" represents the divisor,
"q" represents the quotient,
"R" represents the remainder.

This division relation may be expressed as follows:

$$a = (b \cdot q) + R$$

Consider the following example in division:

$$\begin{array}{rr} & 24 \text{ Remainder } 23 \\ 25 & \overline{)623} \\ & \underline{50} \\ & 123 \\ & \underline{100} \\ & 23 \end{array}$$

We can write this problem in the form

$$623 = (25 \times 24) + 23.$$

This follows the general form:

$$\text{dividend} = (\text{divisor} \times \text{quotient}) + \text{remainder}$$

or

$$a = (b \cdot q) + R$$

Exercises 5-5

1. Copy and complete the following table. Do this carefully as you will use the table in answering Question 2.

	DIVIDEND =	(DIVISOR ·	QUOTIENT) +	REMAINDER
EXAMPLE	9	4	2	1
a.	12	6	?	0
b.	14	3	?	?
c.	29	?	3	2
d.	37	5	?	?
e.	38	9	?	?
f.	41	13	?	?
g.	59	?	5	9
h.	?	11	6	0
i.	77	?	3	17
j.	81	?	?	0

2. Use the table in Problem 1 in answering parts a , b, and c.
   a. Compare the divisor and quotient in each part. Does one of these always have the greater value in a division problem?
   b. Compare the quotient and dividend. Which one has the greater value, if the dividend and divisor are both counting numbers?
   c. Compare the divisor and the remainder. Which one always has the greater value in a division problem?
   d. Can the dividend be zero? Is so, give an example.

   - e. Can the divisor be zero? If so, give an example.
   - f. Can the quotient be zero? If so, give an example.
   - g. Can the remainder be zero? If so, give an example.

3. Using the table in Problem 1, answer the following questions.
   - a. Can any whole number appear as a dividend? If not, give an example.
   - b. Can any whole number appear as a divisor? If not give an example.
   - c. Can any whole number appear as a quotient? If not give an example.
   - d. Must the remainder always be some whole number? Explain.
4. Copy and complete the following table for the division relation.

$$a = (b \cdot q) + R$$

	a	b	q	R
a.	15	?	7	?
b.	?	10	9	8
c.	50	12	?	?
d.	100	?	?	0
e.	283	17	?	?
f.	630	?	25	5

5. Using the table above answer the following:
   - a. Can R be greater than b? If so, give an example.
   - b. Can q be greater than b? If so, give an example.
   - c. Can R be greater than the quotient q? Is so, give an example.
   - d. Can any whole number be a possible value of b? Explain.
   - e. Can any counting number be a possible value of b? Explain.
   - f. Can any whole number be a possible value of a? Explain.
6. Using the division relation, $a = (b \cdot q) + R$, where $R < b$, answer the following:
   - a. If $b = 4$, write the set of all possible remainders.

b. If $b = 11$, describe the set of all possible remainders.

c. If all the possible remainders in a division problem are the whole numbers less than 25, what is b?

d. If $b = K$, which one of the following represents the number of all possible remainders.

(K), $(K + 1)$, or $(K - 1)$.

*7. In Chapter 5, Section 4, we learned how to find the greatest common factor of two numbers. By using the division relation we have another method for doing this.

EXAMPLE: (A). Find the greatest common factor of 12 and 8.

(1) First, divide the larger number by the smaller:
12 divided by 8 = 1 Remainder 4

(2) Second, divide the divisor, 8, by the remainder, 4:
8 divided by 4 = 2 Remainder 0

(3) The 4 is the last divisor used which gives a remainder of 0. The greatest common factor of 8 and 12 is 4.

EXAMPLE: (B). Find the greatest common factor of 35 and 56.

(1) First, divide the larger number, 56, by the smaller number, 35.

$$\begin{array}{rl} 1 & \text{Remainder } 21 \\ 35\,\overline{)\,56} & \\ \underline{35} & \\ 21 & \end{array}$$

(2) Second, divide the divisor, 35, by the remainder 21.

$$\begin{array}{rl} 1 & \text{Remainder } 14 \\ 21\,\overline{)\,35} & \\ \underline{21} & \\ 14 & \end{array}$$

(3) Next, continue dividing the last divisor by the last remainder until the remainder is 0.

$$\begin{array}{rl} 1 & \text{Remainder } 7 \\ 14\,\overline{)\,21} & \\ \underline{14} & \\ 7 & \end{array} \qquad \begin{array}{rl} 2 & \text{Remainder } 0 \\ 7\,\overline{)\,14} & \end{array}$$

The last divisor used is the greatest common factor. The 7 is the greatest common factor of 35 and 56. Note that when 14 is divided by 7 the remainder is 0. The 7 is the last divisor used.

Using the above method, find the greatest common factor for each of the following pairs of numbers:

a. 32 and 92
b. 81 and 192
c. 72 and 150
*d. 124 and 836
*e. 336 and 812
*f. 1207 and 1349

5-6. Review

1. Do the computations indicated. Check all problems after part a, using the inverse operation.
   a. $13 + 729 + 206 + 48$
   b. $500 - 399$
   c. 57×89
   d. $4269 \div 3$
   e. $1325 - 764$
   f. 308×47
   g. $3612 \div 4$
   h. 2344×601
   i. $445 - 366$
   j. $8301 \div 5$
2. Perform the following arithmetic operations. Show work and answers in base seven numerals.
   a. $21_{seven} + 34_{seven}$
   b. $352_{seven} + 416_{seven}$
   c. $416_{seven} - 352_{seven}$
   d. $306_{seven} - 155_{seven}$
3. List the set of all common factors for each of the following:
   a. 18, 42
   b. 21, 33
   c. 24, 60
   d. 39, 78
4. List, in base seven numerals, the set of all common factors for each of the following:
   a. 30_{seven}, 50_{seven}
   b. 42_{seven}, 60_{seven}
   c. 66_{seven}, 51 seven
   *d. 100_{seven}, 100_{ten}
5. Do the computations indicated. Check all problems after part a, using the inverse operation.
   a. $985 + 726 + 673 + 1548$
   b. $90{,}703 - 70{,}309$ check

c. 60,004 - 54,927 check

d. 237 × 405 check

e. 32,396 ÷ 89 check

*f. 167,544 ÷ 276 check

**g. 14,411 ÷ 2401 check

6. Find the greatest common factor of the numbers given in each of the following parts:

   a. 18, 42
   b. 28, 56
   c. 48, 84
   d. 29, 92

7. Find the greatest common factor for each of the following sets. Write your answers in base seven notation.

   a. 15_{seven}, 33_{seven}
   b. 26_{seven}, 50_{seven}
   c. 66_{seven}, 110_{seven}

8. Perform the following arithmetic computations. Express all answers in lowest terms.

   a. $\frac{3}{8} + \frac{3}{8}$
   b. $\frac{8}{9} \times \frac{3}{4}$
   c. $8 \div \frac{2}{5}$
   d. $\frac{15}{16} - \frac{7}{16}$
   e. $\frac{10}{21} - \frac{5}{21}$
   f. $\frac{7}{9} - \frac{4}{9}$

9. Consider three lines in a plane.

   a. If no two of the lines are parallel, how many points of intersection do the three lines have? Draw figures to illustrate your answer.
   b. How many points of intersection do the three lines have if only two of the lines are parallel?
   c. How many points of intersection do the three lines have if all the lines are parallel?

10. Assume you are in a room whose walls, floor and ceiling are all rectangular in shape.

    a. How many examples can you find where three planes suggested by the walls (or floor or ceiling) intersect in a point?

b. How many examples can you find where two such planes intersect in a line?

11. Which of the following are divisible by 2?

a. 101_{two}

b. 101_{five}

c. 101_{six}

d. 101_{seven}

e. 101_{eight}

f. 101_{ten}

12. Which of the following are multiples of 3?

a. 12_{three}

b. 13_{four}

c. 14_{five}

d. 15_{seven}

e. 17_{eight}

f. 18_{nine}

13. List each of the following under one of these two headings:

Composite Numbers   Prime Numbers

a. 24_{five}

b. 111_{two}

c. 155_{seven}

d. 91

e. 63

f. 10_{two}

g. 103

h. 35_{nine}

14. a. Draw two angles such that the intersection of their interiors is the interior of a triangle.

b. Draw two rays whose union is a line.

15. Write in words the numbers represented below:

a. 700,003

b. 803,040

c. 610,502

d. 129,047

16. a. Write the set of all possible remainders in a division problem, if the divisor is 5.

b. What is the divisor if the largest possible remainder in a division problem is 13?

c. Write the set of all possible factors of 31. Explain why there are only two members in this set.

17. Write the numbers represented by the following, using numerals:

a. eight hundred one

b. three hundred seven thousand

c. twenty one thousand twenty four

d. $4\frac{1}{2}$ million

*18. a. Is the set of prime numbers closed under the addition operation?

b. Is the set of prime numbers closed under the multiplication operation?

*19. Suppose a, b, f, and g are four different numbers. They are all multiples of f. The greatest common factor of a and b is g. Using one of the symbols ($<$, $=$, or $>$), describe the relationship between each of the following.

a. f $\underline{?}$ a

b. f $\underline{?}$ b

c. f $\underline{?}$ g

d. Write a number for each of the letters to illustrate these relationships.

*20. Assume the numbers in Question 19 are not necessarily different. Using the symbols ($<$, $=$, or $>$), describe the possible relationships for the following.

a. f $\underline{?}$ g

b. a $\underline{?}$ f

c. g $\underline{?}$ b

d. a $\underline{?}$ b

*21. a. The factors of 7 are 1 and 7. Which property of whole numbers permits us to write $1 \times 7 = 7 \times 1$?

b. A complete factorization of 12 is $12 = 2 \times 2 \times 3$ or $12 = 2^2 \times 3$. Which two properties of whole numbers permits us to write $2 \times 2 \times 3 = 2 \times 3 \times 2 = 3 \times 2 \times 2$?

c. Except for order, what is the greatest possible number of sets of prime factors of any one given counting number.

*22. The numerals from 1 through 50 are written on separate sheets of paper and mixed up in a hat. If one piece of paper is drawn from the hat and then returned each time, what is the chance that the numeral drawn represents a number divisible by each of the following?

a. 10 b. 5 c. 2 d. 1

*23. Refer to Question 22. What is the least number of pieces of paper you must draw from the hat to be assured of having a numeral divisible by each of the following?

a. 1 b. 2 c. 3 d. 5

*24. Is there a one-to-one correspondence between the following set of numbers? If so, set up the correspondence.

a. The set of counting numbers from 1 through 11 and the set of whole numbers from 0 through 10.

b. The set of odd numbers between 50 and 80 and the set of even numbers between 17 and 47.

c. The set of all multiples of 3 which are less than 44 and the set of all multiples of 7 which lie between 100 and 200.

5-7. Least Common Multiple

You have already learned a great deal about multiples of numbers:

that all whole numbers are multiples of 1;
that even numbers {0, 2, 4, 6, ...,} are multiples of 2;
that {0, 3, 6, 9, ...,} are multiples of 3.

Similarly we can list the multiples of any counting number.

The number 2 is an even number, and the number 3 is an odd number. Usually we do not think of such numbers as having much in common. Yet if we look at the set of multiples for 2 and the set of multiples of 3 we see that they do have something in common. Some of the multiples of 2 are also multiples of 3. For example, 6 is a multiple of both 2 and 3. There are many such numbers divisible by both 2 and 3. The set of these numbers is written as follows:

{6, 12, 18, 24, 30, ...,}

Definition: Numbers which are multiples of more than one number are called common multiples of those numbers. "Common" means belonging to more than one. Thus 6 and 12 are common multiples of 2 and 3.

Let's try another example. List the common multiples of 3 and 4. First, we list the multiples of each:

Set of multiples of 3: {0, 3, 6, 9, 12, 15, 18, 21, 24, ...,}
Set of multiples of 4: {0, 4, 8, 12, 16, 20, 24, ...,}

The numbers that these sets have in common are the common multiples of 3 and 4. This set is written as follows:

$$\{0, 12, 24, 36, 48, \ldots,\}$$

This set is the intersection of the two previous sets.

Common multiples are very useful in arithmetic. For example, let us add $\frac{1}{2} + \frac{1}{3}$. We write $\frac{1}{2}$ as $\frac{3}{6}$ and $\frac{1}{3}$ as $\frac{2}{6}$. Then $\frac{3}{6} + \frac{2}{6} = \frac{3 + 2}{6} = \frac{5}{6}$. Here we use a common multiple of 2 and 3. In doing such problems you may have called the 6 a "common denominator." It is a common multiple of the denominators of the given fractions.

Since 6, 12, 18, and so on, are multiples of 2 and 3, we can use any of these numbers in adding $\frac{1}{2} + \frac{1}{3}$. Notice that the number, 6, which we did use is the smallest of those possible. It is also the smallest of the common multiples of 2 and 3. The number, 6, is called the least common multiple of 2 and 3.

Definition: The least common multiple of a set of counting numbers is the smallest counting number which is a multiple of each member of the set of given numbers.

Note that 0 is a common multiple for any set of whole numbers However, in adding or subtracting fractions, 0 cannot be used as a common denominator. Can you write $\frac{1}{2}$ with a zero denominator? Because we cannot do so, we are interested only in the least common multiple other than zero.

Suppose we wish to find the least common multiple of 12 and 18. First, we list the sets of multiples of each:

Set of Multiples of 12: $\{0, 12, 24, 36, 48, 60, 72, 84, \ldots,\}$
Set of Multiples of 18: $\{0, 18, 36, 54, 72, \ldots,\}$

The set of common multiples of 12 and 18 is $\{0, 36, 72, 108, \ldots,\}$. The smallest counting number in this set is 36. Therefore, 36 is the least common multiple of 12 and 18.

What is the least common multiple of 2, 3, and 4?

Set of Multiples of 2: {0, 2, 4, 6, 8, 10, 12, ...,}
Set of Multiples of 3: {0, 3, 6, 9, 12, 15, ...,}
Set of Multiples of 4: {0, 4, 8, 12, 16, 20, ...,}

The set of common multiples of 2, 3, and 4 is: {0, 12, 24, 36,...,. What is the smallest counting number in this set? According to our definition, the least common multiple of 2, 3, and 4 is 12.

Exercise 5-7

1. Write the set of all multiples less than 100 for each of the following.
   a. 6
   b. 8
   c. 9
   d. 12
2. Using your answers in Problem 1, write the set of all common multiples, less than 100 for each of the following.
   a. 6 and 8.
   b. 6 and 9
   c. 6 and 12
   d. 8 and 9
   e. 8 and 12
   f. 9 and 12
3. Using your answers in Problem 2, write the least common multiple of the elements of each of the following sets.
   a. 6 and 8
   b. 6 and 9
   c. 6 and 12
   d. 8 and 9
   e. 8 and 12
   f. 9 and 12
4. Find the least common multiple of the elements of each of the following sets:
   a. {2, 5}
   b. {4, 6}
   c. {2, 3, 5}
   d. {3, 4, 6}
   e. {2, 5, 6}
   f. {4, 5, 6}
   g. {2, 6, 7}
   h. {8, 9, 12}

5. Find the least common multiple of the elements of the following sets:
   - a. {2, 3}
   - b. {3, 5}
   - c. {3, 7}
   - d. {5, 7}
   - e. {2, 11}
   - f. {5, 11}
   - g. {2, 13}
   - h. {7, 11}
   - i. {3, 13}
   - j. {11, 13}
   - k. {2, 3, 5}
   - l. {23, 29}

6. Refer to Problem 5 and answer the following questions:
   - a. To which set do the numbers 2, 3, 5, 7, 11, 13, 23, and 29 belong -- the set of composite numbers or prime numbers?
   - b. From your answers in problem 5, what appears to be an easy way to find the least common multiple in those cases?

7. Find the least common multiple for each of the following sets:
   - a. {4, 6}
   - b. {4, 8}
   - c. {4, 10}
   - d. {6, 9}
   - e. {8, 10}
   - f. {10, 12}
   - g. {12, 15}
   - h. {4, 6, 10}
   - i. {10, 15, 30}
   - j. {4, 6, 8}

8. In Problem 7, to which set of numbers, composite or prime, do each of the numbers, 4, 6, 8, ..., in parts a through j belong?

9. Compare the questions and your answers in Problems 7 and 8. Then answer the following:
   - a. If c and d are composite counting numbers can c or d be the least common multiple? Write an example to explain your answer.
   - b. If c and d are composite counting numbers, must c or d be the least common multiple? Write an example to your answer.

10. a. What is the least common multiple of 6 and 6?
    - b. What is the least common multiple of 29 and 29?
    - c. What is the least common multiple of a and a where a is any counting number?

11. a. What is the least common multiple of 1 and 6?
    b. What is the least common multiple of 1 and 29?
    c. What is the least common multiple of 1 and a where a represents any counting number?

12. a. If a and b are different prime numbers, can a or b represent the least common multiple of a and b?
    b. If a and b are different prime numbers how can we represent the least common multiple of a and b?
    *c. If a, b, and c are different prime numbers, what is the least common multiple of a, b, and c?

13. Study the following examples. Try to discover a shorter way to determine the least common multiple.

EXAMPLE A: To find the least common multiple of 4, 6, and 8:

(1) First, write a complete factorization for each number.
$4 = 2^2$ $6 = 2 \cdot 3$ $8 = 2^3$

(2) The least common multiple is $2^3 \cdot 3$ or 24.

(3) Note that $2^2 \cdot 2 . 3 \cdot 2^3 = 192$ which is a common multiple of 4, 6, and 8, but not the least.

EXAMPLE B: To find the least common multiple of 12 and 18.

(1) A complete factorization for each number:
$12 = 2^2 \cdot 3$ $18 = 2 \cdot 3^2$

(2) The least common multiple of 12 and 18 is $2^2 . 3^2$ or 36.

(3) Is $(2^2 \cdot 3 \cdot 2 \cdot 3^2)$ a common multiple of 12 and 18?

(4) Is $(2^2 \cdot 3 \cdot 2 \cdot 3^2)$ the least common multiple of 12 and 18?

Now find the least common multiple of each set in the following parts.

a. 12, 16
b. 14, 16
c. 9, 15
d. 10, 14
e. 16, 18
f. 4, 5, 6

g.	6, 8, 9	*k.	250, 200
h.	8, 9, 10	*l.	324, 144, 180
i.	12, 20, 22	*m.	306, 1173
j.	9, 16, 20		

*14. a. Is there a greatest common multiple of 3 and 5? If so, write an example.

b. Is there a greatest common multiple of 4 and 6? If so, write an example.

c. Is there a greatest common multiple of any set of counting numbers?

*15. a. May we consider 0 as a multiple of zero? (Does $0 \times 0 = 0$?)

b. May we consider 0 as a multiple of six? (Does $6 \times 0 = 0$?)

c. May we consider 0 as a multiple of a if a is any whole number?

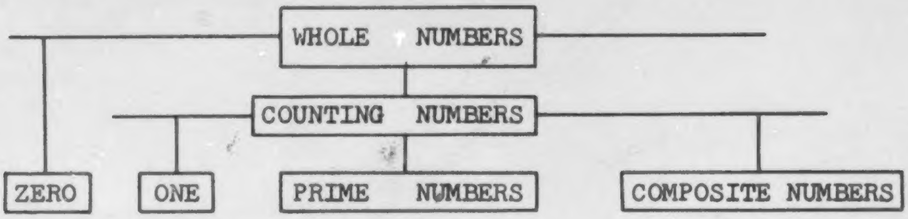
d. Assume the least common multiple was defined as "the smallest whole number" instead of "the smallest counting number." What would be the least common multiple for any set of counting numbers?

e. Using the correct definition for least common multiple, is there a least common multiple for any counting number and 0?

5-8. Summary

In this chapter you studied whole numbers for the most part. Also, you have studied some important subsets of whole numbers.

These subsets are shown in the sketch below:



> Note that zero is a member of the set of whole numbers, but not a member of the set of counting numbers. The ONE, the PRIME NUMBERS, and the COMPOSITE NUMBERS are members of the set of COUNTING NUMBERS and also members of the set of WHOLE NUMBERS.
>
> Every member of the set of counting numbers is a member of the set of whole numbers.

You learned that a PRIME number is any counting number, other than 1, that is divisible only by itself and 1. The number 1 is not a prime number. We chose not to include 1 as a prime number because any number can be expressed as the product of primes in many different ways if we include 1 in the set of prime numbers.

A COMPOSITE number is a counting number, other than 1, that is not prime. Composite numbers have more than two factors.

The term "factor" was used instead of the words multiplicand and multiplier. The number, a, is a FACTOR of b if b is divisible by a. The set of factors of a number contains all counting numbers which are factors. A COMPLETE FACTORIZATION of a number represents the number as a product of prime numbers. For a prime number this is the number itself. For a composite number there are two or more factors. The UNIQUE FACTORIZATION PROPERTY of counting numbers refers to the fact that every composite number can be expressed as the product of primes in only one way, except for order.

A COMMON FACTOR of a set of whole numbers is a number that is a factor of each member of the set of numbers. The GREATEST COMMON FACTOR of a set of whole numbers is the largest counting number which is a factor of each member of the set of numbers. A common factor can never be greater than the largest member of the set.

The whole number, b, is a MULTIPLE of the whole number, a, if $a \cdot c = b$, where c is also a whole number. A COMMON MULTIPLE of a set of numbers is a multiple of each member of the set of numbers. The LEAST COMMON MULTIPLE is the smallest counting number which is a multiple of every member of the set of numbers. The least common multiple cannot be less than the largest member of the set of numbers.

Exercises 5-8

1. Find the greatest common factor of the numbers in each of the following sets of numbers.
   a. {2, 3}
   b. {6, 8}
   c. {7, 14}
   d. {15, 25}
   e. {12, 36}
   f. {15, 21}
   g. {23, 43}
   h. {66, 78}
   i. {39, 51}
   *j. {74, 146}
   *k. {45, 72, 252}
   **l. {44, 92, 124}
2. Find the least common multiple of the numbers in each of the sets of numbers in parts a. through l. in Problem 1.
3. a. Find the product of the members of each set of numbers in Problem 1.
   b. Find the product of the greatest common factor and the least common multiple for each set of numbers in problem one. (Refer to your answers for Problem 1 and Problem 2.)
   c. How do your answers for a and b compare?
4. a. Write the set of all composite numbers less than 31.
   b. Write the set of all prime numbers less than 51.

5. Let a and b represent two counting numbers. Suppose that the greatest common factor of a and b is 1.
   a. What is the least common multiple of a and b? Give an example to explain your answer.
   b. Would your answer for part a. be true if you started with three counting numbers a, b, and c? (Remember, the greatest common factor is 1.) Give an example to explain your answer.
6. a. Can a prime number be even? Give an example to explain your answer.
   b. Can a prime number be odd? Give an example to explain your answer.
   c. How many prime numbers end with the digit 5?
   d. With the exception of two prime numbers, all primes end with one of four digits. Write the two primes which are exceptions.
   e. Write the other four digits which occur in the ones place for all primes other than the exception you found in part d.
7. Suppose the greatest common factor of two numbers is the same as their least common multiple. What must be true about the numbers? Give examples to explain your answer.
8. a. What is the least common factor of 2867 and 6431?
   b. What is the greatest common multiple of 2867 and 6431?
9. 112 tulip bulbs are to be planted in a garden. Describe all possible arrangements of the bulbs if they are to be planted in straight rows with an equal number of bulbs per row.
10. Two bells are set so that their time interval for striking is different. Assume that at the beginning both of the bells strike at the same time.
    a. One bell strikes every three minutes and the second strikes every five minutes. If both bells strike together at 12:00 o'clock noon, when will they again strike together?

[sec. 5-8]

b. One bell strikes every six minutes and the second bell every fifteen minutes. If both strike at 12:00 o'clock noon, when will they again strike together?

c. Find the least common multiple of 3 and 5 and of 6 and 15. How do these answers compare with parts a. and b.?

11. a. Can the greatest common factor of some whole numbers ever be the same number as the least common multiple of those whole numbers? If so, give an example.

b. Can the greatest common factor of some whole numbers ever be greater than the least common multiple of those numbers? If so, give an example.

c. Can the least common multiple for some whole numbers ever be less than the greatest common factor of those whole numbers? If so, give an example.

*12. a. Is it possible to have exactly four composite numbers between two consecutive primes? If so, give an example.

b. Is it possible to have exactly five consecutive composite numbers between two consecutive primes? If so give an example.

*13. Given the numbers 135, 222, 783, 1065. Without dividing answer the following questions. Then check your answers by dividing.

a. Which numbers are divisible by 3?
b. Which numbers are divisible by 6?
c. Which numbers are divisible by 9?
d. Which numbers are divisible by 5?
e. Which numbers are divisible by 15?
f. Which numbers are divisible by 4?

*14. Why is it important to learn about prime numbers?

15. BRAINBUSTER. Ten tulip bulbs are to be planted so that there will be exactly five rows with four bulbs in each row. Draw a diagram of this arrangement.

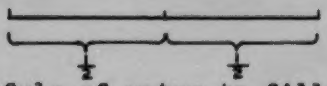
16. BRAINBUSTER. Do you think there is a largest prime number? Can you find it or can you give a reason why you think there is no greatest one?

Chapter 6

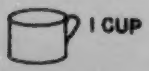
THE RATIONAL NUMBER SYSTEM

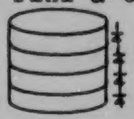
6-1. History of Fractions

Man has not always known about fractions. Historically, he introduced fractions when he began to measure as well as count. If he divided a piece of string into two parts of equal length, then each part was $\frac{1}{2}$ as long as the original string.

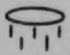


If he had to pour 4 cupfuls of water to fill a container,





then he said that the cup held $\frac{1}{4}$ the amount of water in the container.

The Egyptians worked with fractions. At first they used only unit fractions and the fractions, $\frac{2}{3}$ and $\frac{3}{4}$. Unit fractions are fractions with numerators of 1, such as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, etc. The Egyptians used the notation  for $\frac{1}{5}$, that is, the numeral for 5 with a special mark written over it. When they had to use other fractions, they expressed them in terms of unit fractions:

$$\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$$

$$\frac{15}{24} = \frac{1}{2} + \frac{1}{8}$$

The Rhind Papyrus (1700 BC), copied by the scribe Ahmes from an older document now lost, has a set of tables showing how to express fractions in terms of unit fractions.

The Babylonians usually used fractions with denominators of 60, 60^2 (3600), and 60^3 (216,000), etc. because the base of their system of notation was 60. Since we borrowed our units of time from the Babylonians, we also divide an hour into sixtieths, called minutes, and a minute into sixtieths, called seconds.

Roman children primarily learned about fractions with denominators of twelve. They did not have symbols for fractions but they did have names for fractions such as $\frac{1}{12}$, $\frac{2}{12}$, ... Do we use any measures today that use the idea of twelfths?

Over the years many other notations were used. Our present notation with the fraction bar "—" came into general use in the 16th century.

6-2. Rational Numbers

In the rest of this chapter we are going to look at fractions in a new way. You already know a great deal about fractions. Are you sure you know *why* your methods work?

Reasons are important. Do you sometimes ask "Why?" when your parents tell you something. Knowing *why* helps you to understand your parents' rules. You may find it easier to remember how to work with fractions when you know *why* the rules work.

You have learned that a number may have several names. Several names or numerals for the same number are 6, VI, $2 \cdot 3$, $3 \cdot 2$. Some other names for this number are the fractions $\frac{6}{1}$, $\frac{12}{2}$, $\frac{18}{3}$, $\frac{24}{4}$. Each fraction is a name for a certain number. The fractions $\frac{1}{3}$, $\frac{2}{6}$, $\frac{3}{9}$, $\frac{4}{12}$, $\frac{5}{15}$, $\frac{6}{18}$ represent one number. The fractions $\frac{3}{2}$, $\frac{6}{4}$, $\frac{9}{6}$, $\frac{12}{8}$, $\frac{15}{10}$, $\frac{18}{12}$ represent another number. Such numbers are called *rational numbers*. Numerals for other rational numbers are $\frac{7}{1}$, $\frac{8}{2}$, $\frac{3}{4}$. What are some other names for these numbers?

How do you check the division problem:

$$12 \div 3 = 4?$$

You multiply 3 by 4 to see if you get 12. When you divide 12 by 3 you are finding an answer to the question:

3 times what number equals 12?

or

$$3 \cdot ? = 12$$

You will find it better to use a letter such as "x" instead of "?" for the number you are seeking, so write

$$3 \cdot x = 12.$$

If you replace x by 4, the number sentence,

$$3 \cdot x = 12$$

is true. Mathematicians usually write $\frac{12}{3}$ instead of $12 \div 3$. So they write, $\frac{12}{3} = 4$.

With our new symbol for division we write,

$$\frac{12}{3} = 4 \text{ because } 3 \cdot 4 = 12,$$

$$\frac{6}{2} = 3 \text{ because } 2 \cdot 3 = 6,$$

$$\frac{10}{2} = 5 \text{ because } 2 \cdot 5 = 10,$$

$$\frac{63}{9} = 7 \text{ because } 9 \cdot 7 = 63.$$

Because $\frac{12}{3} = 4$ we can replace 4 by $\frac{12}{3}$ in $3 \cdot 4 = 12$ and write

$$3 \cdot \frac{12}{3} = 12.$$

Similarly,

$$2 \cdot \frac{6}{2} = 6 ,$$

$$2 \cdot \frac{10}{2} = 10,$$

$$9 \cdot \frac{63}{9} = 63.$$

By what number should x be replaced to make the number sentence

$$3 \cdot x = 12$$

true? Yes, x = 4. But if you look back in the last paragraph you will see that we wrote

$$3 \cdot \frac{12}{3} = 12.$$

Both 4 and $\frac{12}{3}$ are correct replacements for x. The numerals 4 and $\frac{12}{3}$ stand for the same number. Likewise,

if $x = \frac{6}{2}$, then $2 \cdot x = 6$,

if $x = \frac{7}{3}$, then $3 \cdot x = 7$.

Class Discussion Problems.

1. (a) If $x = \frac{10}{2}$, $2 \cdot x$ is what number?

   (b) If $x = \frac{63}{9}$, $9 \cdot x$ is what number?

   (c) If $x = \frac{5}{2}$, $2 \cdot x$ is what number?

   (d) If $x = \frac{5}{3}$, $3 \cdot x$ is what number?

   (e) If $x = \frac{4}{9}$, $9 \cdot x$ is what number?

2. For each of the following give a fractional name for the number represented by x.

   (a) $2 \cdot x = 10$, (f) $6 \cdot x = 5$,

   (b) $9 \cdot x = 63$, (g) $4 \cdot x = 13$,

   (c) $2 \cdot x = 5$, (h) $7 \cdot x = 1$,

   (d) $3 \cdot x = 5$, (i) $4 \cdot x = 2$,

   (e) $9 \cdot x = 4$, (j) $1 \cdot x = 13$.

3. If a and b are counting numbers, give a fractional name for the number represented by x in $b \cdot x = a$.

In general, if a and b are whole numbers, and b is not zero, $\frac{a}{b}$ is the number x for which $b \cdot x = a$. Because it is simpler, a symbol such as $b \cdot x$ is usually written bx, and $8 \cdot x$ is written 8x. The multiplication symbol is still necessary in writing a symbol such as $8 \cdot 6$. Why? You have heard that division is the inverse of multiplication. Here we have used this to change the question $\frac{a}{b} = ?$
to $b \cdot ? = a$.

A symbol " $\frac{a}{b}$ " where a and b are numbers, with b not zero, is called a fraction. If a and b are whole numbers, with b not zero, the number represented by the fraction, $\frac{a}{b}$, is called a rational number; any number which can be written in this form is called a rational number. For example, 0.5 represents a rational number because the same number can be written $\frac{1}{2}$. A fraction is a name for a rational number just as numeral is a name for a number. Different names for the same number are:

$$3\ ,\ \text{III}\ ,\ \frac{6}{2}\ ,\ \frac{9}{3}\ ,\ \frac{63}{21}\ .$$

The names

$$\frac{6}{2}\ ,\ \frac{9}{3}\ ,\ \frac{63}{21}$$

are fractions.

Sometimes $\frac{a}{b}$ is a whole number. This happens when b is a factor of a, and only then.

Sometimes $\frac{a}{b}$ is not a whole number. Is there a whole number for which $3x = 4$? Is $\frac{4}{3}$ a whole number?

Two fractions which represent the same number are called equivalent fractions. But it will not often be necessary to use this term.

Exercises 6-2

1. Give an example of each of the following kinds of numbers.
    (a) Counting number.
    (b) Whole number.
    (c) A whole number which is not a counting number.
    (d) A rational number which is not a whole number.

2. Which of the following represent rational numbers?

(a) $\frac{3}{7}$	(e) 1
(b) $\frac{5}{3}$	(f) $\frac{16}{4}$
(c) $\frac{1}{10}$	(g) 0.2
(d) 4	(h) 0.13

3. Copy and complete the following statements.

   (a) If $x = \frac{6}{3}$, then $3x =$ ____ .

   (b) If $x = \frac{9}{3}$, then ___$x = 9$.

   (c) If $x = \frac{5}{2}$, then $2x =$ ____ .

   (d) If $x = \frac{10}{4}$, then ___$x = 10$.

   (e) If $x = \frac{7}{1}$, then ___$x = 7$.

   (f) If $x = \frac{18}{6}$, then ___$x =$ ____ .

   (g) If $x = \frac{4}{3}$, then ___$x =$ ____ .

4. For problem 3, in which cases is the number x a whole number? Whenever x is a whole number write it with a single digit.

5. Copy and complete the following statements.

   (a) If $x = \frac{1}{5}$, then ___$x =$ ___ .

   (b) If $x = \frac{4}{4}$, then ___$x =$ ___ .

   (c) If $x = \frac{11}{3}$, then ___$x =$ ___ .

   (d) If $x = \frac{63}{9}$, then ___$x =$ ___ .

   (e) If $x = \frac{0}{5}$, then ___$x =$ ___ .

   (f) If $x = \frac{123}{11}$, then ___$x =$ ___ .

6. For problem 5, in which cases is the number x a whole number? When x is a whole number, write it with a single digit.

7. Without dividing or factoring, decide which of the following statements are true. As an example, to show that $\frac{168}{21} = 8$, multiply 8 by 21 to see if you get 168.

   (a) $\frac{169}{13} = 13$

   (b) $\frac{262}{17} = 16$

   (c) $\frac{744}{124} = 6$

   (d) $\frac{143}{11} = 13$

   (e) $\frac{15251}{151} = 101$

*8. For each of the following, write a number sentence which describes the problem in mathematical language. Use x for the unknown number, and tell, in each case, for what it stands.

   Example: Sam's father is sawing a twelve-foot log into 6 equal lengths. How long will each piece be?

   Answer: If x is the length of each piece in feet, then $6 \cdot x = 12$.

   (a) If 12 cookies are divided equally among 3 boys, how many cookies does each boy receive?

   (b) Mr. Carter's car used 10 gallons of gasoline for a 160-mile trip. How many miles did he drive for each gallon of gasoline used?

   (c) If it takes 20 bags of cement to build a 30-foot walk, how much cement is needed for each foot of the walk?

   (d) Thirty-two pupils were divided into 4 groups of the same size. How many pupils were in each group?

   (e) A teacher has 12 sheets of paper to distribute evenly in a class of 24. How much paper will each pupil receive?

6-3. Properties of Rational Numbers

You have seen that the whole number 3 can be written, $\frac{3}{1}$, which shows that 3 is a rational number. In a similar way, you can show that each whole number is a rational number.

When you studied whole numbers you learned that the whole numbers had certain properties. Learning about rational numbers is made easier by knowing that the rational numbers have some of the same properties.

You remember that the sum of two whole numbers is always a whole number, and the product of two whole numbers is always a whole number. That is, if a and b are whole numbers, there is a whole number c for which $a + b = c$ and a whole number d for which $a \cdot b = d$. The set of whole numbers has the closure property for addition and multiplication.

The set of rational numbers also has the closure property for addition and multiplication. The sum of two rational numbers is a rational number. You know that $\frac{2}{5} + \frac{1}{5} = \frac{3}{5}$, $\frac{12}{4} + \frac{1}{4} = \frac{13}{4}$, $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$, $2 + \frac{1}{4} = \frac{9}{4}$. The product of two rational numbers is a rational number. Notice that $\frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$, $\frac{2}{3} \cdot \frac{6}{7} = \frac{12}{21}$, $\frac{5}{1} \cdot \frac{1}{6} = \frac{5}{6}$. In precise language we state:

1) The set of rational numbers is closed with respect to the operations of addition and multiplication.

You know that $3 + 4 = 4 + 3$ and $3 \cdot 4 = 4 \cdot 3$ because, for the whole numbers, addition and multiplication have the commutative property. These operations also have the commutative property for the rational numbers. You know that $\frac{2}{5} + \frac{3}{5} = \frac{3}{5} + \frac{2}{5}$, and $\frac{2}{5} \cdot \frac{3}{5} = \frac{3}{5} \cdot \frac{2}{5}$. In precise language we state:

2) The operations of addition and multiplication for the rational numbers have the commutative property, that is:

$$a + b = b + a \quad \text{and} \quad a \cdot b = b \cdot a$$

You also remember that $5 + (3 + 4) = (5 + 3) + 4$, and $5 \cdot (3 \cdot 4) = (5 \cdot 3) \cdot 4$, because addition and multiplication have the associative property for the whole numbers. For the rational numbers also, these operations have the associative property. You know that $\frac{5}{3} + (\frac{1}{3} + \frac{2}{3}) = (\frac{5}{3} + \frac{1}{3}) + \frac{2}{3}$, and that $\frac{3}{1} \cdot (\frac{1}{3} \cdot \frac{6}{3}) = (\frac{3}{1} \cdot \frac{1}{3}) \cdot \frac{6}{3}$. In precise language, we state:

3) The operations of addition and multiplication for the rational numbers have the associative property, that is:

$$a + (b + c) = (a + b) + c \text{ , and } a(bc) = (ab)c \text{ .}$$

What is $5 \cdot (2 + 3)$? From the distributive property you know that you get the same result if you think of $5 \cdot 5 = 25$ as you do if you think of $5 \cdot 2 + 5 \cdot 3 = 10 + 15 = 25$. For the whole numbers, multiplication is distributive over addition. The distributive property also holds for the rational numbers. You have used this property for the rational numbers when you multiplied $4\frac{1}{5}$ by 5. In our symbols, $5 \cdot (4\frac{1}{5}) = 5 \cdot (4 + \frac{1}{5}) = 5 \cdot 4 + 5 \cdot \frac{1}{5} = 20 + 1 = 21$. In precise language, we state:

4) The operation of multiplication is distributive over addition for the rational numbers; that is:

$$a(b + c) = ab + ac$$

Among the whole numbers were two special numbers 1 and 0. These are also rational numbers.

5) Among the rational numbers are special numbers 0 and 1; 0 is the identity for addition and 1 is the identity for multiplication.

When we say that 0 is the identity for addition we mean, for instance, that $0 + 3 = 3 + 0 = 3$; that is, that adding zero to any number does not change it. This can be expressed in symbols as:

$$0 + a = a + 0 = a \text{ ,}$$

no matter what number a is. Similarly when we say that 1 is the identity for multiplication, we mean, for instance, that $1 \cdot 5 = 5 \cdot 1 = 5$; that is, multiplying any number by 1 does not change it. This can be expressed in symbols as:

$$1 \cdot a = a \cdot 1 = a .$$

You can see that 1 is a rational number by writing it as the fraction $\frac{1}{1}$. To see that 0 is a rational number you should remember that 0 divided by any counting number is 0. If $x = \frac{0}{1}$, $1 \cdot x = 0$ and x must equal zero. In defining a rational number, $\frac{a}{b}$, we said that b could not be zero. You can see the reason for this by seeing what happens to $\frac{5}{0}$.

If $x = \frac{5}{0}$, then $0 \cdot x = 5$. There is no number x for which $0 \cdot x = 5$ so there is no number $\frac{5}{0}$.

These five properties of the rational numbers let you see the reasons for some of the rules you state for fractions. Let us use these properties to show $\frac{3}{2} = \frac{15}{10}$. If $x = \frac{3}{2}$, then $2x = 3$. Since $2x$ and 3 are names for the same number,

$$5 \cdot (2x) = 5 \cdot 3 .$$

By the associative property,

$$(5 \cdot 2)x = 5 \cdot 3$$
$$10\,x = 15$$
$$x = \frac{15}{10}.$$

But x is a name for $\frac{3}{2}$, so

$$\frac{3}{2} = \frac{15}{10}$$

If you write the last equation,

$$\frac{3}{2} = \frac{5 \cdot 3}{5 \cdot 2} ,$$

you see that you would have arrived at the same fraction if you had multiplied the numerator and denominator of $\frac{3}{2}$ by 5. Generalizing, we get

Property 1. If the numerator and denominator of a fraction are multiplied by the same counting number, the number represented is not changed. If the numerator and denominator are divided by the same counting number, the number represented is not changed.

You saw that $\frac{3}{2}$ and $\frac{15}{10}$ are fractions for the same number. Other names for this number are: $\frac{6}{4}$, $\frac{9}{6}$, $\frac{18}{12}$. By what numbers should you multiply the numerator and denominator of $\frac{3}{2}$ to get these fractions? Since in $\frac{3}{2}$ the numerator and denominator have no common factors except 1, this is called the *simplest form of the fraction*.

To find the simplest form of $\frac{72}{45}$ we find the greatest common factor of 72 and 45, which is 9. Then

$$\frac{72}{45} = \frac{9 \cdot 8}{9 \cdot 5} = \frac{8}{5}\,.$$

You may prefer to take more steps and do it this way:

$$\frac{72}{45} = \frac{3 \cdot 24}{3 \cdot 15} = \frac{24}{15} = \frac{3 \cdot 8}{3 \cdot 5} = \frac{8}{5}\,.$$

To write the fraction $\frac{a}{b}$ in simplest form, we find the greatest common factor k of a and b, where $a = kc$ and $b = kd$; then by Property 1,

$$\frac{a}{b} = \frac{kc}{kd} = \frac{c}{d}\,.$$

Exercises 6-3

1. Which of the following represent rational numbers?

   (a) 7

   (b) $\frac{6}{2}$

   (c) $\frac{9}{5}$

   (d) $3 + \frac{1}{2}$

   (e) $4\frac{1}{3}$

   (f) $4 - 2$

   (g) $1 + 5$

   (h) 2.15

   (i) $\frac{0}{7}$

   (j) 5.0

   (k) $\frac{5}{0}$

   (l) 0.0

2. Write each of these numbers in three different ways.

(a) $\frac{1}{2}$

(b) $\frac{2}{5}$

(c) $\frac{3}{3}$

(d) $\frac{12}{4}$

(e) $0 \cdot 2$

(f) $\frac{12}{6}$

(g) $\frac{0}{5}$

(h) $\frac{25}{30}$

(i) $\frac{30}{25}$

(j) 6

3. Write in simplest form.

(a) $\frac{2}{6}$

(b) $\frac{12}{15}$

(c) $\frac{30}{8}$

(d) $\frac{14}{21}$

(e) $\frac{8}{2}$

(f) $\frac{48}{16}$

(g) $\frac{13}{39}$

(h) $\frac{40}{50}$

(i) $\frac{500}{400}$

(j) $\frac{17}{51}$

4. Each of the following is true by one of the properties of rational numbers. Name the property in each case.

(a) $\frac{2}{3} \cdot \frac{5}{6} = \frac{5}{6} \cdot \frac{2}{3}$

(b) $\frac{5}{2}(\frac{3}{2} \cdot \frac{2}{5}) = (\frac{5}{2} \cdot \frac{3}{2}) \cdot \frac{2}{5}$

(c) $5(2 + 3) = 5 \cdot 2 + 5 \cdot 3$

(d) $\frac{1}{2}(4 + 6) = \frac{1}{2}(4) + \frac{1}{2}(6)$

(e) $1 \cdot \frac{5}{6} = \frac{5}{6}$

(f) $\frac{2}{3} \cdot \frac{5}{5} = \frac{2}{3}$

(g) $0 + \frac{7}{8} = \frac{7}{8}$

(h) $\frac{3}{2} + \frac{4}{5} = \frac{4}{5} + \frac{3}{2}$

(i) $\frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) = (\frac{1}{2} + \frac{1}{3}) + \frac{1}{4}$

(j) $5(6 \cdot 7) = (5 \cdot 6) \cdot 7$

5. Carry out the indicated operations in each of the following:

   (a) $\frac{8 + 6}{2}$ (b) $\frac{91 - 35}{7}$

   (c) $\frac{60 \cdot 3}{9 \cdot 2}$ (d) $\frac{51 - 21}{7 + 3}$

6. Three dollars are divided equally among 4 people.
   (a) How much will each person receive?
   (b) Use a fraction to express the part of a dollar each person will receive.

7. Which would you prefer to receive, one of eight equal parts of five dollars, or five of the eight equal parts of one dollar?

8. Draw diagrams and shade parts of the diagrams to show:
   (a) three of the 5 equal parts of one unit,
   (b) one of the 5 equal parts of three units.
   (c) Represent the shaded part of your diagrams in (a) and (b) by a rational number.
   (d) How could your diagrams in (a) and (b) be thought of as representing the commutative property of multiplication?

9. A man owns a ranch which contains 6 square miles of land. He makes a will dividing the ranch equally among 8 sons. How many square miles of land will each son receive?

10. What is the difference between 53 of the 67 equal parts of a rectangle, and one of the 67 equal parts of 53 rectangles, assuming that all rectangles considered are equal in size?

*11. Find five pairs of counting numbers which can be used as values for n and d to make the following number sentence true:

$$3n = 2d.$$

Compute $\frac{n}{d}$ in each case. What is the general rule?

*12. Use the number sentence $2x = 3$ to show that $\frac{3}{2} = \frac{1}{2} \cdot 3$. Hint: If $2x$ and 3 are names for the same number then $\frac{1}{2}(2x) = \frac{1}{2} \cdot 3$.

6-4. Reciprocals

You know that

$$2 \cdot \frac{1}{2} = 1, \quad 3 \cdot \frac{1}{3} = 1, \quad \frac{1}{4} \cdot 4 = 1, \quad \frac{1}{5} \cdot 5 = 1, \quad 31 \cdot \frac{1}{31} = 1.$$

Let us recall our definition of a rational number and see how these products are related to our definition.

a and b are
whole numbers,
$b \neq 0$

$$bx = a$$
$$x = \frac{a}{b}$$

Let $a = 1$
Let $b = 31$

1 and 31 are
whole numbers,
$31 \neq 0$

$$31x = 1$$
$$x = \frac{1}{31}$$

We know that $\frac{1}{31}$ is the rational number by which we can multiply 31 to obtain 1.

By what number may we replace x in

$$24x = 1$$

so that we have a true sentence? The answer is $\frac{1}{24}$.

Now let us consider the equation

$$bx = 1.$$

Then $$x = \frac{1}{b} \quad \text{(b is a counting number.)}$$

and $$b \cdot \frac{1}{b} = 1.$$

The number $\frac{1}{b}$ is called the reciprocal of b. Also, b is called the reciprocal of $\frac{1}{b}$. *If the product of two numbers is 1, the numbers are called reciprocals of each other.*

Some pairs of reciprocals are:

$$\frac{1}{7} \text{ and } 7, \quad 21 \text{ and } \frac{1}{21}, \quad \frac{1}{62} \text{ and } 62.$$

What is the reciprocal of 1?

We have seen that counting numbers and numbers like $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{29}$ have reciprocals. Does $\frac{3}{4}$ have a reciprocal? Is there a number by which $\frac{3}{4}$ may be multiplied to get 1? You may know the answer from arithmetic.

We know that $\frac{3}{4} \cdot 4 = 3$ and $3 \cdot \frac{1}{3} = 1$, so

$$\begin{aligned} \frac{3}{4} \cdot (4 \cdot \frac{1}{3}) &= (\frac{3}{4} \cdot 4) \cdot \frac{1}{3} \\ &= (3) \cdot \frac{1}{3} = 1 , \end{aligned}$$

using the associative property.

We see that if $\frac{3}{4}$ is multiplied by $4 \cdot \frac{1}{3}$ the product is 1. But $4 \cdot \frac{1}{3} = \frac{4}{3}$. Hence, the reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$.

What is the product $\frac{7}{8} \cdot \frac{8}{7}$? Why? We can see why the product is 1, by recalling that $\frac{8}{7} = 8 \cdot \frac{1}{7}$,

$$\frac{7}{8} \cdot \frac{8}{7} = \frac{7}{8} \cdot (8 \cdot \frac{1}{7}) = (\frac{7}{8} \cdot 8) \cdot \frac{1}{7} = 7 \cdot \frac{1}{7} = 1.$$

From your experience in multiplying numbers represented by fractions you probably know, without showing all the steps, that $\frac{8}{7} \cdot \frac{7}{8} = 1$. The steps show why this is a true sentence, making use of the properties of rational numbers that we know.

The examples lead us to the conclusion that $\frac{a}{b} \cdot \frac{b}{a} = 1$, provided neither a nor b is zero.

Property 2. The reciprocal of the rational number $\frac{a}{b}$ is the rational number $\frac{b}{a}$, if $a \neq 0$ and $b \neq 0$.

Exercises 6-4

1. Find the products.

(a) $(9)(\frac{1}{9})$

(b) $(\frac{1}{26})(26)$

(c) $(\frac{1}{11})(11)$

(d) $(45)(\frac{1}{45})$

(e) $(\frac{3}{5})(5)(\frac{1}{3})$

(f) $92(\frac{1}{51})$

(g) $(\frac{7}{4})(\frac{1}{7})(7)$ (i) $(\frac{9}{11})(\frac{11}{9})$

(h) $(\frac{4}{5})(3)$ (j) $(\frac{35}{4})(\frac{4}{35})$

2. Write the reciprocals of the rational numbers.

(a) 11 (e) $\frac{2}{7}$

(b) 201 (f) $\frac{50}{3}$

(c) 47 (g) $\frac{1000}{7}$

(d) $\frac{1}{5}$ (h) $\frac{346}{175}$

3. In the following, letters represent rational numbers, all different from zero. Write the reciprocals.

(a) m (b) s (c) $\frac{1}{c}$ (d) $\frac{r}{s}$ (e) $\frac{t}{w}$

4. Find x so that these sentences are true:

(a) $3x = 5$ (c) $15x = 19$

(b) $7x = 2$ (d) $36x = 18$

(e) $rx = k,\ r \neq 0$

5. Write the following as sentences involving multiplication, and find n in each.

(a) $8 \div 7 = n$ (e) $1492 \div 13 = n$

(b) $15 \div 20 = n$ (f) $6 \div 300 = n$

(c) $100 \div 17 = n$ (g) $5 \div 475 = n$

(d) $2 \div 11 = n$ (h) $64 \div 36 = n$

6. Write the set of numbers consisting of the reciprocals of the members of the set, Q, where

$$Q = \{1, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}\}.$$

7. (a) When is the reciprocal of a number greater than the number?
   (b) When is the reciprocal of a number less than the number?
   (c) When is the reciprocal of a number equal to the number?
   (d) If n is a counting number, can we correctly say that one of the following is always true?

$$(1)\ n > \frac{1}{n},\quad (2)\ \frac{1}{n} > n,\quad (3)\ n = \frac{1}{n}$$

8. The product of what number and n is 1, where n is a number such that $2n = 19$?

9. (a) Find n if $8n = 36$.
   (b) Find q if $36q = 8$.
   (c) What can you say about the numbers n and q?

10. The population of Cary is 16,000 and of a neighboring city, Davis, 30,000.
    (a) The population of Davis is how many times the population of Cary?
    (b) The population of Cary is how many times the population of Davis?
    (c) What can you say about the answers for (a) and (b)?

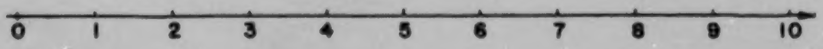
*11. If $14n = 5$, then $(\frac{1}{5})(14n) = (\frac{1}{5})(5)$. (a) Why? (b) Use the equation in (a) to find the number by which we can multiply n (given by $14n = 5$) to obtain 1.

*12. If $ax = b$ and $by = a$, and a and b are not zero, x and y are reciprocals. Why is this true?

6-5. Using The Number Line

Recall how the number line was constructed in chapter 3. We started with a line and selected a point which we called "0".

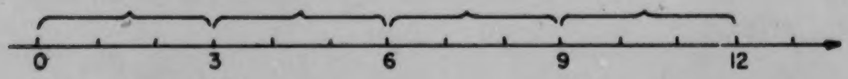
After selecting a unit of length, we marked off this distance on the ray, starting at 0 and extending to the right. After doing this again and again we can label the endpoints of the segments 1, 2, 3, and so on, and have the picture shown below:



The counting number at each point shows how many segments of unit length are measured off from 0 to that point.

To add 3 and 2 on the number line, we start with 0 and lay off a distance of 3 equal units. Then we measure off 2 more equal units to the right. The union of these two segments is that whose left end is at 0 and whose right end is at 5; the length of this segment is the sum of the lengths of the other two segments. Thus we have a picture on the line which shows the sum: $3 + 2 = 5$. Can you see how to subtract on the number line?

To multiply 3 by 4 on the number line we can start from 0 and lay off four segments of length 3 as shown below:



This also gives a means of division. If we divide the segment with endpoints 0 and 12 into four equal parts, each part will be of length 3. Hence the same picture can represent both

$$12 = 4 \times 3 \quad \text{and} \quad \frac{12}{4} = 3.$$

If we think of 12 as designating 12 feet, the line shows that there are four yards in twelve feet and hence

$\frac{12}{3} = 4$, or, twelve-thirds is equal to four.

In words: "Twelve feet is equal to four yards" may also be written "twelve thirds-of-a-yard is equal to four yards," since one foot is one third-of-a-yard.

The same number line can also be used to represent the number of yards in, say, 13 feet. Here the segment can be laid off four times with a segment of length 1 foot left over. Then, since one foot is one third of a yard, we have

$$13 \cdot \frac{1}{3} = \frac{13}{3} = 4 + \frac{1}{3}$$

which can be read as follows: thirteen feet is equal to four yards and one third of a yard. This same equality can also be written in the form

$$13 = 4 \cdot 3 + 1,$$

which can be read: 13 feet is equal to four yards plus one foot.

Suppose we consider another example and seek to represent on the number line the number of feet in 43 inches. Here you should draw the number line showing the equation:

$$43 = 3 \cdot 12 + 7,$$

which can be read: 43 inches is equal to three feet and 7 inches. Using fractions this could also be written

$$\frac{43}{12} = 43 \cdot \frac{1}{12} = 3 + \frac{7}{12}$$

expressing the fact that 43 twelfths-of-a-foot is equal to 3 feet plus 7 twelfths-of-a-foot.

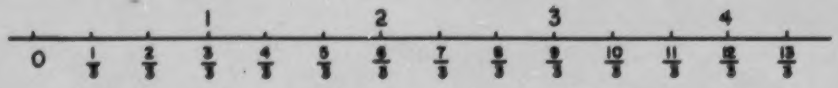
Though the number line helps to show these relationships in the beginning, it becomes rather awkward if the numbers involved are larger. Suppose we consider the fraction $\frac{85}{23}$ without constructing the number line. If we divide 85 by 23, we have the quotient 3 and the remainder 16, that is,

$$85 = 23 \cdot 3 + 16$$

Another way of writing this is:

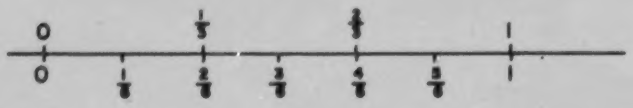
$$\frac{85}{23} = 3 + \frac{16}{23} = 3\frac{16}{23} .$$

So far we have shown on the number line only segments whose lengths are whole numbers of units. We could just as well have considered fractions by labeling the points differently. For instance, we might have:



We could see from this just as well as from the previous number line that $12 \cdot \left(\frac{1}{3}\right) = 4$. Also we can see that $\frac{13}{3} = 4 + \frac{1}{3}$ from the same number line.

Recall that Property 1, near the end of Section 6-3, expressed the fact that we can multiply the numerator and denominator of a fraction by a counting number without changing the number which it represents. If the denominator is small, we can show this on the number line. For instance, $\frac{1}{3} = \frac{2}{6}$ can be seen from the following:



Property 1 gives us a way to simplify certain fractions. It can also be used to show whether or not the numbers represented by two fractions are equal.

Example 1. Suppose we have the question:

$$\text{Is } \frac{6}{9} = \frac{8}{12} \text{ ?}$$

This can be answered by finding the simplest form of each of the fractions as follows:

$$\frac{6}{9} = \frac{2 \cdot 3}{3 \cdot 3} = \frac{2}{3} \quad \text{and} \quad \frac{8}{12} = \frac{2 \cdot 4}{3 \cdot 4} = \frac{2}{3} .$$

Since each number is equal to $\frac{2}{3}$, the given numbers are equal to each other.

Example 2. Is $\frac{9}{15}$ equal to $\frac{20}{25}$? By the same method we have

$$\frac{9}{15} = \frac{3 \cdot 3}{5 \cdot 3} = \frac{3}{5} \quad \text{and} \quad \frac{20}{25} = \frac{4 \cdot 5}{5 \cdot 5} = \frac{4}{5}.$$

Since $\frac{3}{5}$ is not equal to $\frac{4}{5}$ the given fractions do not represent the same number.

Example 3. Is $\frac{9}{15}$ equal to $\frac{14}{22}$? Now,

$$\frac{9}{15} = \frac{3}{5} \quad \text{and} \quad \frac{14}{22} = \frac{7 \cdot 2}{11 \cdot 2} = \frac{7}{11} .$$

Is $\frac{3}{5}$ equal to $\frac{7}{11}$? Is the answer as clear as it was in the first two examples? Whatever your answer is, can you give a reason? We found that it is easier to compare fractions if they have the same denominator. This suggests another method which we may use to answer the question, "Is $\frac{3}{5}$ equal to $\frac{7}{11}$?" We can rewrite the two fractions with a common denominator, that is, having the same denominator. This denominator must be a multiple of both 5 and 11. The least such denominator is 55, since this is the least common multiple of 5 and 11. Then

$$\frac{3}{5} = \frac{3 \cdot ?}{5 \cdot 11}$$

shows that since we multiplied the denominator by 11 to get $5 \cdot 11$, we must also multiply the numerator by 11, that is, we must also fill in the question mark with an 11. Thus

$$\frac{3}{5} = \frac{3 \cdot 11}{5 \cdot 11} = \frac{33}{55}$$

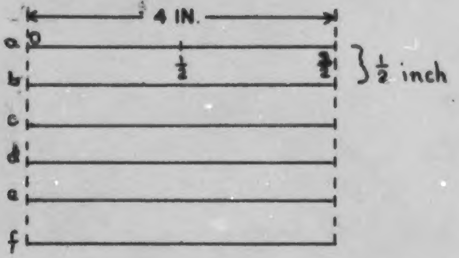
and similarly

$$\frac{7}{11} = \frac{7 \cdot 5}{11 \cdot 5} = \frac{35}{55} .$$

Are $\frac{33}{55}$ and $\frac{35}{55}$ equal? You know that 33 divided by 55 is not the same number as 35 divided by 55. Therefore $\frac{9}{15}$ is not equal to $\frac{14}{22}$.

Exercises 6-5

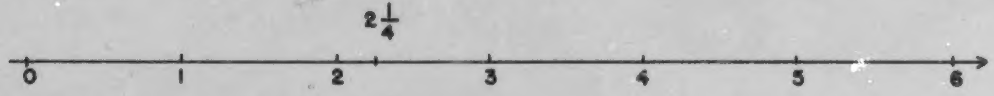
1. On a sheet of paper, draw a figure similar to the one shown below. Make each line segment 4 inches long and allow $\frac{1}{2}$ inch between lines.



Divide each segment into the following number of parts:

(a) 2 parts (Shown on drawing above.)
(b) 4 parts
(c) 8 parts
(d) 3 parts
(e) 6 parts
(f) 5 parts

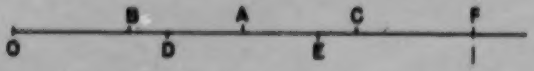
2. On a sheet of paper make a drawing of a number line like this:



Use a unit length of 1 inch. As closely as you can, locate each of the following points on your number line as shown for (a):

(a) $2\frac{1}{4}$
(b) $4\frac{2}{3}$
(c) $\frac{10}{2}$
(d) $\frac{15}{3}$
(e) $\frac{14}{3}$
(f) $\frac{19}{8}$
(g) If $\frac{a}{b} = \frac{6}{3}$, locate $\frac{a}{b}$ on the number line.
(h) If $12 \cdot x = 4$, locate x on the number line.

3. (a) Consider the points labeled A, B, C, D, E, and F on the number line:



Give 3 different fraction names for each of the points A, B, C, D, E, and F as:

A: $\frac{1}{2}$, $\frac{2}{4}$, __ ; etc.

(b) Is the rational number located at point B less than or greater than the one located at A? Explain your answer.

(c) Is the rational number located at C less than or greater than that at A? Explain your answer.

4. Interpret on the number line the following:

(a) $\frac{20}{5} = 4$ (b) $\frac{20}{4} = 5$ (c) $\frac{23}{5} = 4\frac{3}{5}$

5. Express each of the following as a whole number added to a rational number less than 1:

(a) $\frac{57}{39}$ (b) $\frac{137}{23}$

6. Show on the number line the equality: $\frac{2}{8} = \frac{3}{12}$

7. Answer the questions in Problems 1 and 2 of this section using the method of Example 3, page 209.

8. Show how to subtract on the number line for each of the following:

(a) 3 - 2 (b) 9 - 5

*9. In Example 3 we found whether two fractions represented the same number by writing each of them as equivalent fractions with equal denominators. Could we also have found whether they represented the same number by writing each of them as equivalent fractions with equal numerators? Remember that we called two fractions "equivalent" if they represented the same number.

6-6. Multiplication of Rational Numbers

You recall that $3 \cdot \frac{1}{3} = 1$ and $3 \cdot \frac{2}{3} = 2$. In general if a and b are whole numbers, where b is not zero,

$$b \cdot \frac{1}{b} = 1 \quad \text{and} \quad b \cdot \frac{a}{b} = a.$$

What does $\frac{5}{7} \cdot 2$ equal? You can find an answer if you can find a number x for which

$$x = \frac{5}{7} \cdot 2.$$

But then, x and $\frac{5}{7} \cdot 2$ are names for the same number, so that

$$7x = 7(\frac{5}{7} \cdot 2).$$

Using the associative property for multiplication

$$7x = (7 \cdot \frac{5}{7}) \cdot 2 = 5 \cdot 2 = 10.$$

$$x = \frac{5 \cdot 2}{7} = \frac{10}{7}.$$

You started with $x = \frac{5}{7} \cdot 2$, and you have shown that $x = \frac{5 \cdot 2}{7}$, so

$$\frac{5}{7} \cdot 2 = \frac{5 \cdot 2}{7}.$$

In general, if a, b, and c are whole numbers and b is not zero,

$$c \cdot \frac{a}{b} = \frac{c \cdot a}{b}.$$

Similarly, you can find a number x for which

$$x = \frac{2}{3} \cdot \frac{5}{7}.$$

Since x and $\frac{2}{3} \cdot \frac{5}{7}$ are names for the same number

$$3x = 3(\frac{2}{3} \cdot \frac{5}{7}).$$

Using the associative property,

$$3x = (3 \cdot \frac{2}{3}) \cdot \frac{5}{7}$$

$$3x = 2 \cdot \frac{5}{7}$$

$$3x = \frac{2 \cdot 5}{7}.$$

But $3x$ and $\frac{2 \cdot 5}{7}$ are names for the same number. Thus

$$7 \cdot (3x) = 7(\frac{2 \cdot 5}{7}).$$

$$7 \cdot (3x) = 2 \cdot 5.$$

Using the associative property for multiplication,

$$(7 \cdot 3)x = 2 \cdot 5.$$

$$x = \frac{2 \cdot 5}{7 \cdot 3} = \frac{2 \cdot 5}{3 \cdot 7} = \frac{10}{21}.$$

You started with $x = \frac{2}{3} \cdot \frac{5}{7}$, and you have shown that $x = \frac{2 \cdot 5}{3 \cdot 7}$, so

$$\frac{2}{3} \cdot \frac{5}{7} = \frac{2 \cdot 5}{3 \cdot 7}.$$

This example shows that:

To multiply two rational numbers written as fractions, you find a fractions whose numerator is the product of the numerators and whose denominators is the product of the denominators.

In symbols, if $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers,

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} = \frac{ac}{bd}.$$

You can use this property to reduce $\frac{12}{16}$ to simplest form, instead of Property 1, in this way

$$\frac{12}{16} = \frac{4 \cdot 3}{4 \cdot 4} = \frac{4}{4} \cdot \frac{3}{4} = 1 \cdot \frac{3}{4} = \frac{3}{4}.$$

Similarly,

$$\frac{21}{35} = \frac{7 \cdot 3}{7 \cdot 5} = \frac{7}{7} \cdot \frac{3}{5} = 1 \cdot \frac{3}{5} = \frac{3}{5}.$$

You can even show Property 1 for the rational number $\frac{a}{b}$ this way if k is not zero:

$$\frac{a}{b} = 1 \cdot \frac{a}{b} = \frac{k}{k} \cdot \frac{a}{b} = \frac{k \cdot a}{k \cdot b} = \frac{ka}{kb}.$$

Exercises 6-6

1. Find the following products mentally. Give answers in simplest form.

(a) $\frac{1}{2} \cdot \frac{1}{3}$ (b) $\frac{1}{5} \cdot \frac{2}{3}$ (c) $\frac{3}{4} \cdot \frac{1}{5}$

(d) $\frac{2}{5} \cdot \frac{3}{5}$ (e) $\frac{1}{10} \cdot \frac{3}{10}$ (f) $\frac{4}{100} \cdot \frac{3}{10}$

(g) $\frac{1}{9} \cdot 9$ (h) $\frac{2}{9} \cdot 2$ (i) $2 \cdot \frac{2}{9}$

(j) $4 \cdot \frac{1}{3}$ (k) $\frac{1}{3} \cdot \frac{1}{3}$ (l) $\frac{3}{2} \cdot 4$

(m) $8 \cdot \frac{3}{4}$ (n) $(\frac{1}{2} \cdot \frac{1}{4}) \cdot \frac{1}{3}$ (o) $\frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10}$

(p) $10 \cdot \frac{1}{100}$ (q) $\frac{1}{100} \cdot \frac{1}{100}$ (r) $\frac{1}{10} \cdot \frac{1}{100}$

(s) $\frac{1}{10} \cdot 100$ (t) $\frac{5}{2} \cdot 8$ (u) $\frac{5}{6} \cdot \frac{3}{3} \cdot \frac{1}{5}$

(v) $\frac{6}{5} \cdot \frac{5}{6}$ (w) $\frac{21}{2} \cdot \frac{1}{2}$ (x) $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$

2. Find the products in simplest form.

(a) $\frac{1}{3} \cdot \frac{1}{13}$ (b) $\frac{1}{8} \cdot \frac{1}{15}$ (c) $\frac{1}{12} \cdot \frac{1}{5}$

(d) $\frac{1}{5} \cdot 50$ (e) $\frac{2}{3} \cdot \frac{1}{8}$ (f) $\frac{5}{6} \cdot 24$

(g) $36 \cdot \frac{1}{12}$ (h) $\frac{4}{7} \cdot \frac{1}{12}$ (i) $72 \cdot \frac{5}{9}$

(j) $\frac{8}{9} \cdot 1$ (k) $\frac{4}{5} \cdot \frac{17}{17}$ (l) $\frac{9}{16} \cdot \frac{7}{10}$

(m) $\frac{13}{15} \cdot \frac{10}{39}$ (n) $\frac{33}{100} \cdot \frac{90}{99}$ (o) $\frac{17}{3} \cdot \frac{3}{51}$

3. Often factoring may make the computation of products like those in Problem 2 easier. For example in part (e)

$$\frac{2}{3} \cdot \frac{1}{8} = \frac{2 \cdot 1}{3 \cdot 8} = \frac{2 \cdot 1}{3 \cdot (2 \quad 4)} = \frac{2 \cdot 1}{2 \cdot 3 \cdot 4} = \frac{2}{2} \cdot \frac{1}{3 \cdot 4} = \frac{1}{12} = \frac{1}{12}.$$

Use this method for parts h, m, n, and o of Problem 2.

4. Find each product as a fraction in simplest form. Before you multiply, express each part of the product as a single fraction. For example $\frac{3}{4} \cdot (2\frac{1}{2}) = \frac{3}{4} \cdot \frac{5}{2} = \frac{15}{8}$.

(a) $8 \cdot (2\frac{2}{3})$ (b) $(3\frac{1}{3}) \cdot (2\frac{2}{3})$

(c) $12 \cdot (5\frac{5}{6})$ (d) $(2\frac{1}{5}) \cdot (3\frac{1}{3})$

(e) $(5\frac{1}{4}) \cdot (\frac{2}{7})$ (f) $(1\frac{1}{7}) \cdot (1\frac{1}{8})$

5. Find the number r for which the following statement is true.

$$\frac{3}{4} \cdot (\frac{1}{3} \cdot \frac{7}{8}) = (r \cdot \frac{1}{3}) \cdot \frac{7}{8}.$$

6. The label on a can of paint states that the can contains $\frac{3}{4}$ of a pint. If you use $\frac{2}{3}$ of the paint, how many pints did you use?

*7. Here is a recipe for fruit punch to serve 6 persons:

$\frac{1}{2}$ cup orange juice

$\frac{1}{4}$ cup lemon juice

$\frac{1}{2}$ cup grapefruit juice

1 cup pineapple juice
$1\frac{1}{2}$ cups water
$\frac{1}{2}$ cup syrup

(a) What amount of each ingredient would you use to make punch for 4 persons?

(b) What amount of each ingredient would you use to serve 8 persons?

*8. On a road map 1 inch represents 10 miles. If the distance on this map from your house to school measures $1\frac{5}{8}$ inches, how many miles do you live from school?

6-7. Division of Rational Numbers

In this chapter you learned that you could write $5 \div 3$ as $\frac{5}{3}$. If a and b are any two numbers where b is not zero, you can write $\frac{a}{b}$ instead of $a \div b$. If a and b are rational numbers, $\frac{3}{2}$ and $\frac{1}{2}$, for example, you can write

$$\frac{3}{2} \div \frac{1}{2} = \frac{\frac{3}{2}}{\frac{1}{2}} .$$

Notice that the bar between $\frac{3}{2}$ and $\frac{1}{2}$ is longer than the other two bars.

When you studied inverse operations you learned that multiplication and division were inverse operations. This means that

$$\frac{8}{2} = 4 \text{ if } 2 \cdot 4 = 8$$

and only then. This same relation holds between multiplication and division if a, b, and x are rational numbers, and b is not zero. That is,

$$\frac{a}{b} = x \quad \text{if} \quad bx = a$$

and only then. For example

$$\frac{\frac{3}{2}}{\frac{1}{2}} = x \quad \text{if} \quad \frac{1}{2}x = \frac{3}{2}$$

and only then. Here $x = 3$ makes both statements true. Other names for x are $\frac{3}{1}$ and $\frac{6}{2}$.

As another example, look at

$$x = \frac{\frac{3}{2}}{\frac{5}{7}} .$$

How do you find x? You can find x without using rules for division.

$$x = \frac{\frac{3}{2}}{\frac{5}{7}} \qquad \text{if} \qquad \frac{5}{7}x = \frac{3}{2}$$

and only then. In the right hand equality $\frac{5}{7}x$ and $\frac{3}{2}$ are both names for the same number. Now multiply this number by $\frac{7}{5}$, the reciprocal of $\frac{5}{7}$, and use the associative property to get

$$\left(\frac{7}{5} \cdot \frac{5}{7}\right) \cdot x = \frac{7}{5} \cdot \frac{3}{2}$$

$$1 \cdot x = \frac{7}{5} \cdot \frac{3}{2}$$

$$x = \frac{7}{5} \cdot \frac{3}{2} .$$

But at the beginning of the example you had

$$x = \frac{\frac{3}{2}}{\frac{5}{7}} ,$$

so

$$\frac{\frac{3}{2}}{\frac{5}{7}} = \frac{7}{5} \cdot \frac{3}{2} .$$

Using only rules we have had earlier we have shown in the example the reason for the following statement.

> To find the quotient of two rational numbers written as fractions you find the product of the numerator and the reciprocal of the denominator. In symbols:
>
> $$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} .$$

It is not even necessary to remember this statement about dividing one rational number by another if you divide in the following way.

$$\frac{\frac{3}{2}}{\frac{5}{7}} = \frac{\frac{3}{2} \cdot \frac{7}{5}}{\frac{5}{7} \cdot \frac{7}{5}} = \frac{\frac{3}{2} \cdot \frac{7}{5}}{1} = \frac{3}{2} \cdot \frac{7}{5} .$$

You merely have to look at the denominator and decide what number you must multiply it by to get the number 1, then multiply numerator and denominator by that number. Here is another example.

$$\frac{\frac{4}{3}}{5} = \frac{\frac{4}{3} \cdot \frac{1}{5}}{5 \cdot \frac{1}{5}} = \frac{\frac{4}{3} \cdot \frac{1}{5}}{1} = \frac{4}{3} \cdot \frac{1}{5} = \frac{4}{15} .$$

In an earlier chapter you learned that the whole numbers were not closed for the operation of division. In this section we have shown a method for finding the quotient of any two rational numbers, a and b, b not zero. The operation of division does not allow dividing by zero. Is the set of rational numbers, with zero omitted, closed for the operation of division?

Exercises 6-7

1. Express each of the following as a ratio of two whole numbers.

(a) $\frac{1}{2} \div \frac{1}{2}$

(b) $\frac{1}{3} \div \frac{3}{1}$

(c) $\frac{4}{1} \div \frac{1}{4}$

(d) $12 \div \frac{1}{6}$

(e) $\frac{1}{10} \div 10$

(f) $\frac{5}{9} \div \frac{5}{9}$

(g) $\frac{5}{9} \div \frac{9}{5}$

(h) $\frac{1}{100} \div 10$

(i) $\frac{5}{8} \div \frac{5}{4}$

(j) $\frac{11}{24} \div \frac{11}{2}$

(k) $\frac{1}{3} \div \frac{1}{2}$

(l) $\frac{1}{2} \div \frac{1}{3}$

2. Write each of the following quotients in simplest form. Factor numbers whenever it is to your advantage.

(a) $\frac{\frac{1}{100}}{\frac{1}{10}}$

(b) $\frac{\frac{1}{4}}{\frac{1}{3}}$

(c) $\frac{\frac{1}{9}}{8}$

(d) $\frac{\frac{9}{16}}{\frac{1}{4}}$

(e) $\frac{\frac{1}{3}}{\frac{1}{4}}$

(f) $\frac{\frac{9}{8}}{\frac{1}{2}}$

(g) $\frac{\frac{12}{17}}{6}$

(h) $\frac{\frac{12}{15}}{\frac{24}{30}}$

(i) $\frac{16}{\frac{1}{4}}$

3. Find the following quotients. Express each answer as a whole number, or as a whole number plus a number smaller than 1.

(a) $\frac{1}{4} \div \frac{1}{8}$

(b) $15 \div \frac{1}{5}$

(c) $\frac{2}{3} \div \frac{1}{12}$

(d) $\frac{1}{2} \div \frac{1}{3}$

(e) $\frac{2}{3} \div \frac{1}{4}$

(f) $\frac{3}{4} \div \frac{1}{3}$

(g) $5 \div \frac{2}{3}$

(h) $7 \div 4$

(i) $3 \div \frac{1}{2}$

4. Find the following quotients in simplest form. Start by putting each fraction in simplest form before dividing.

(a) $\dfrac{\frac{1}{9}}{\frac{8}{8}}$

(b) $\frac{3}{11} \div \frac{5}{5}$

(c) $\frac{4}{4} \div \frac{5}{5}$

(d) $\frac{2}{3} \div \frac{6}{8}$

(e) $\frac{4}{6} \div \frac{12}{16}$

(f) $\frac{11}{22} \div \frac{9}{12}$

(g) $\frac{6}{10} \div \frac{8}{12}$

(h) $\frac{120}{60} \div \frac{7}{14}$

(i) $\frac{10}{100} \div \frac{3}{4}$

*5. Do you think division is commutative? Find each quotient to see if you are right.

(a) $\frac{2}{3} \div \frac{1}{2}$

(b) $\frac{1}{2} \div \frac{2}{3}$

*6. Do you think that division is associative? Find each quotient to see if you are right.

(a) $\frac{3}{2} \div (\frac{9}{4} \div \frac{7}{6})$

(b) $(\frac{3}{2} \div \frac{9}{4}) \div \frac{7}{6}$

*7. (a) $\frac{12}{10}$ is how many times as large as $\frac{12}{100}$?

(b) $\frac{4}{7}$ is how many times as large as $\frac{7}{4}$?

6-8. Addition and Subtraction of Rational Numbers

We have examined the elements of the set of rational numbers. We know that there are many names for the same rational number. We have used the operations of multiplication and division. Only two operations remain to be considered: addition and its inverse operation subtraction. Let us look at the addition operation first.

We are all familiar with the idea that $\frac{1}{3} + \frac{1}{3} = 2 \cdot \frac{1}{3} = \frac{2}{3}$; also $\frac{4}{3} = 4 \cdot \frac{1}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$. Continuing with other rational numbers, let us find a single fraction for:

$$\frac{2}{3} + \frac{4}{3}.$$

Using what you already know let us write this as:

$$\frac{2}{3} = 2 \cdot \frac{1}{3} = \frac{1}{3} + \frac{1}{3}$$

and

$$\frac{4}{3} = 4 \cdot \frac{1}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}.$$

Then

$$\frac{2}{3} + \frac{4}{3} = \left(\frac{1}{3} + \frac{1}{3}\right) + \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right) = 6 \cdot \frac{1}{3} = \frac{6}{3}$$

or we may write

$$\begin{aligned} \frac{2}{3} + \frac{4}{3} &= \left(2 \cdot \frac{1}{3}\right) + \left(4 \cdot \frac{1}{3}\right) \\ &= (2 + 4)\frac{1}{3} = 6 \cdot \frac{1}{3} \\ &= \frac{6}{3}, \end{aligned}$$

using the distributive property to get the second line.

What is the result of adding $\frac{a}{b}$ to $\frac{c}{b}$, where a, b, and c are whole numbers and b is not 0? $\frac{a}{b} + \frac{c}{b}$ may be written $a \cdot \frac{1}{b} + c \cdot \frac{1}{b}$. By the distributive property this is equal to

$$(a + c) \cdot \frac{1}{b} = \frac{a + c}{b}.$$

If a, b, and c are whole numbers and b is not 0, then $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$. This may be read in words: The sum of two numbers whose fractions have the same denominator is the sum of the numerators divided by the common denominator.

Let us look at a more difficult addition example:

$$\frac{3}{4} + \frac{7}{10} .$$

The least common multiple of the denominators 4 and 10 is 20. We write each fraction with a denominator 20. You recall that

$$\frac{3}{4} = \frac{3 \cdot 5}{4 \cdot 5} = \frac{15}{20} \text{ and } \frac{7}{10} = \frac{7 \cdot 2}{10 \cdot 2} = \frac{14}{20} .$$

Then

$$\frac{3}{4} + \frac{7}{10} = \frac{15}{20} + \frac{14}{20} = \frac{15 + 14}{20} = \frac{29}{20} .$$

Also

$$\frac{3}{10} + \frac{7}{15} = \frac{3 \cdot 3}{10 \cdot 3} + \frac{7 \cdot 2}{15 \cdot 2} = \frac{9}{30} + \frac{14}{30} = \frac{23}{30} .$$

The sum of any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ may be found similarly. A common multiple of b and d is bd.

$$\frac{a}{b} = \frac{a \cdot d}{b \cdot d} = \frac{ad}{bd} \text{ and } \frac{c}{d} = \frac{b \cdot c}{b \cdot d} = \frac{bc}{bd}$$

Using what we know about adding rational numbers whose fractions have the same denominator we have:

$$\begin{aligned} \frac{a}{b} + \frac{c}{d} &= \frac{ad}{bd} + \frac{bc}{bd} \\ &= \frac{ad + bc}{bd} . \end{aligned}$$

Thus we may say:

If a, b, c, and d are whole numbers and b and d are not zero, then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} .$$

In the following set of exercises you will see again that the commutative and associative properties of addition as well as the distributive property hold in the rational number system.

[sec. 6-8]

Exercises 6-8a

1. Find each of the following sums.

   (a) $\frac{2}{3} + \frac{4}{3}$

   (b) $\frac{4}{3} + \frac{2}{3}$

   (c) $\frac{3}{8} + 4$

   (d) $4 + \frac{3}{8}$

   (e) $\frac{2}{5} + \frac{3}{10}$

   (f) $\frac{3}{10} + \frac{2}{5}$

   (g) $\frac{7}{8} + \frac{3}{16}$

   (h) $\frac{3}{16} + \frac{7}{8}$

2. Does it appear that the addition operation for rational numbers is commutative?

3. Find each of the following sums.

   (a) $(\frac{1}{3} + \frac{2}{3}) + \frac{4}{3}$

   (b) $\frac{1}{3} + (\frac{2}{3} + \frac{4}{3})$

   (c) $(\frac{1}{4} + \frac{5}{4}) + \frac{3}{8}$

   (d) $\frac{1}{4} + (\frac{5}{4} + \frac{3}{8})$

   (e) $(\frac{1}{5} + \frac{5}{2}) + \frac{2}{3}$

   (f) $\frac{1}{5} + (\frac{5}{2} + \frac{2}{3})$

   (g) $(\frac{12}{15} + \frac{6}{3}) + \frac{4}{6}$

   (h) $\frac{12}{15} + (\frac{6}{3} + \frac{4}{6})$

4. Does it appear that the addition operation for rational numbers is associative?

*5. (a) Is it true that $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$ for all rational numbers $\frac{a}{b}$ and $\frac{c}{d}$?

   (b) What property of addition is shown in Part (a)?

*6. (a) Is it true that $(\frac{a}{b} + \frac{c}{d}) + \frac{e}{f} = \frac{a}{b} + (\frac{c}{d} + \frac{e}{f})$ for all rational numbers $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$?

   (b) What principle of addition is shown in Part (a)?

7. (a) A ribbon clerk sold three pieces of ribbons of $\frac{7}{8}$ yard, $\frac{1}{2}$ yard, and $\frac{1}{4}$ yard. How much ribbon did she sell in all?

   (b) If the first and third pieces were ribbon which sells for 25¢ per yard and the second was ribbon which sells for 20¢ per yard, how much was the total cost?

8. In a recipe the ingredients include $\frac{1}{3}$ lb. butter, $\frac{1}{2}$ lb. sugar, $\frac{1}{8}$ lb. cocoa, and $\frac{1}{6}$ lb. peanuts. What is the total weight of these ingredients?

Keeping in mind both the commutative and associative properties of addition we can find the sum of $2\frac{1}{4}$ and $3\frac{5}{8}$ quite easily.

$$\begin{aligned} 2\tfrac{1}{4} + 3\tfrac{5}{8} &= (2 + \tfrac{1}{4}) + (3 + \tfrac{5}{8}) \\ &= (2 + 3) + (\tfrac{1}{4} + \tfrac{5}{8}) \quad \text{by commutative and} \\ &= (2 + 3) + (\tfrac{2}{8} + \tfrac{5}{8}) \quad \text{associative properties.} \\ &= 5 + \tfrac{7}{8} \\ &= 5\tfrac{7}{8} \end{aligned}$$

Does this look familiar to you? You probably arranged your work a little differently but didn't you work similarly? You probably wrote your problem vertically instead of horizontally.

You could also work the preceding problem this way.

$$\begin{aligned} 2\tfrac{1}{4} + 3\tfrac{5}{8} &= \tfrac{9}{4} + \tfrac{29}{8} \\ &= \tfrac{18}{8} + \tfrac{29}{8} \\ &= \tfrac{47}{8} \\ &= 5\tfrac{7}{8} \end{aligned}$$

Exercises 6-8b

1. Find each of the following sums:

(a) $\begin{array}{r} 1\frac{1}{3} \\ + \; 2\frac{2}{3} \\ \hline \end{array}$ (b) $\begin{array}{r} 33\frac{1}{3} \\ + \; 66\frac{2}{3} \\ \hline \end{array}$ (c) $\begin{array}{r} 16\frac{1}{2} \\ + \; 37\frac{1}{3} \\ \hline \end{array}$ (d) $\begin{array}{r} 30 \\ + \; 7\frac{4}{5} \\ \hline \end{array}$

2. Find each of the following sums in two different ways:

(a) $4\frac{1}{8} + 3\frac{4}{8}$ (b) $8\frac{5}{16} + 1\frac{13}{16}$ (c) $2\frac{12}{32} + 14\frac{23}{32}$

3. Find each of the following:

(a) $\frac{1}{32} + \frac{1}{32}$ (g) $(\frac{1}{3} \cdot \frac{1}{2}) + (\frac{1}{3} \cdot \frac{1}{5})$

(b) $1\frac{1}{16} + \frac{10}{16}$ (h) $\frac{1}{3}(\frac{1}{2} + \frac{1}{5})$

(c) $1\frac{2}{7} + \frac{1}{14}$ (i) $(\frac{3}{8} + \frac{6}{9})\ \frac{2}{5}$

(d) $\frac{73}{64} + \frac{20}{32}$ (j) $(\frac{2}{5} \cdot \frac{3}{8}) + (\frac{2}{5} \cdot \frac{6}{9})$

(e) $\frac{10}{21} + 5\frac{2}{3}$ (k) $3\frac{1}{2} + 4\frac{2}{3} + 7\frac{5}{6}$

(f) $\frac{2}{10} + \frac{3}{100} + \frac{1}{1000}$ (l) $\frac{19}{20} + 1\frac{2}{5} + 4\frac{7}{10}$

4. (a) Compare your results in 3(g) and 3(h).
   (b) Compare your results in 3(i) and 3(j).

5. Find the numbers:

(a) $\frac{0}{3} + \frac{7}{8}$ (b) $(3 + 5) + (\frac{1}{3} + \frac{1}{5})$ (c) $\frac{a}{b} + \frac{c}{d}$

6. In each of the following examples, see if you can do the work in your head first. Then check by working out each example on paper.

(a) $1 + (\frac{1}{3} + \frac{1}{5})$ (b) $(1 + \frac{1}{3}) + \frac{1}{5}$

7. Based on your results in 6(a) and 6(b) would you say that division is associative?

8. In the magic square below, add the numbers in each column. Then, adding across, find the sum of the numbers in each row. Now add the numbers in each diagonal. (Top left corner to lower right corner, etc.)

$11\frac{1}{4}$	$2\frac{1}{2}$	$8\frac{3}{4}$
5	$7\frac{1}{2}$	10
$6\frac{1}{4}$	$12\frac{1}{2}$	$3\frac{3}{4}$

9. A man spends $\frac{1}{5}$ of his salary for rent, $\frac{1}{3}$ for food, $\frac{1}{6}$ for clothing, and $\frac{1}{4}$ for charity and service. The rest he saves. Out of each dollar he earns how much does he save?

Since subtraction is so closely related to addition we need not go through all the steps in developing the ideas again. As with addition, if the fractions have the same denominators, then we simply subtract the numerators and place this difference over the common denominator. For example:

$$\frac{12}{7} - \frac{9}{7} = \frac{12 - 9}{7} = \frac{3}{7} .$$

Since subtraction is the inverse operation of addition,

$$\frac{12}{7} - \frac{9}{7} = \frac{3}{7}$$

means the same as

$$\frac{9}{7} + \frac{3}{7} = \frac{12}{7} .$$

In symbols this relation of two rational numbers may be written

$$\frac{c}{b} - \frac{a}{b} = \frac{c - a}{b}$$ where b is not 0 and c is larger than or equal to a.

Remember that the rational numbers of this chapter, are not closed under subtraction.

If two fractions have different denominators, we can rewrite each of them with a common denominator. Let us use this example: $\frac{4}{5} - \frac{5}{7}$.

$$\begin{aligned} \frac{4}{5} - \frac{5}{7} &= \left(\frac{4}{5} \cdot \frac{7}{7}\right) - \left(\frac{5}{7} \cdot \frac{5}{5}\right) \\ &= \frac{28}{35} - \frac{25}{35} \\ &= \frac{28 - 25}{35} \\ &= \frac{3}{35} \end{aligned}$$

Check your answer by addition.

$$\frac{3}{35} + \frac{5}{7} = \frac{4}{5}.$$

As in the operation of addition of rational numbers this same procedure may be applied to the general case $\frac{c}{d} - \frac{a}{b}$.

$$\begin{aligned}\frac{c}{d} - \frac{a}{b} &= \left(\frac{c}{d} \cdot \frac{b}{b}\right) - \left(\frac{a}{b} \cdot \frac{d}{d}\right)\\ &= \frac{cb}{bd} - \frac{ad}{bd}\\ &= \frac{cb - ad}{bd}.\end{aligned}$$

Hence $\frac{c}{d} - \frac{a}{b} = \frac{cb - ad}{bd}$ if (cb) is greater than or equal to (ad) and b and d are not zero.

Class Exercises

1. (a) $\frac{3}{4} - \frac{1}{4}$

(b) $\frac{5}{6} - \frac{4}{6}$

(c) $\frac{9}{8} - \frac{2}{8}$

(d) $\frac{3}{4} - \frac{3}{16}$

(e) $\frac{15}{16} - \frac{5}{8}$

(f) $\frac{11}{12} - \frac{2}{3}$

(g) $\frac{a}{b} - \frac{c}{d}$

(h) $\frac{3}{5} - \frac{2}{7}$

(i) $\frac{23}{100} - \frac{1}{10}$

(j) $\frac{5}{8} - \frac{16}{32}$

(k) $\frac{17}{18} - \frac{1}{9}$

(l) $\frac{1}{100} - \frac{9}{1000}$

(m) $\left(\frac{7}{12} - \frac{4}{12}\right) - \frac{2}{12}$

(n) $\left(\frac{3}{8} + \frac{2}{8}\right) - \frac{3}{8}$

(o) $\left(\frac{2}{9} + \frac{5}{9}\right) - \frac{3}{9}$

2. Express these rational numbers in other ways.

(a) $\frac{9}{8}$, (b) $2\frac{3}{4}$, (c) $\frac{8}{5}$, (d) $5\frac{2}{3}$.

We now know how to subtract any two rational numbers when the result is a rational number. Look at this problem:

$$5\frac{2}{3} - 2\frac{1}{2}$$

$5\frac{2}{3}$ and $2\frac{1}{2}$ may be written in fractional form.

$$\begin{aligned} 5\tfrac{2}{3} - 2\tfrac{1}{2} &= \tfrac{17}{3} - \tfrac{5}{2} \\ &= \left(\tfrac{17 \cdot 2}{3 \cdot 2}\right) - \left(\tfrac{5 \cdot 3}{2 \cdot 3}\right) \\ &= \tfrac{34}{6} - \tfrac{15}{6} \\ &= \tfrac{19}{6} \\ &= 3\tfrac{1}{6} \end{aligned}$$

Should we check by addition here?

You probably learned a little different way of subtracting and would do this same problem in this way:

$$\begin{array}{rcr} 5\frac{2}{3} & = & 5\frac{4}{6} \\ -\ 2\frac{1}{2} & = & -\ 2\frac{3}{6} \\ \hline & & 3\frac{1}{6} \end{array}$$

Checking to see if the answer is correct we find:

$$\begin{aligned} 2\tfrac{1}{2} + 3\tfrac{1}{6} &= 2 + \tfrac{1}{2} + 3 + \tfrac{1}{6} \\ &= (2 + 3) + \left(\tfrac{1}{2} + \tfrac{1}{6}\right) && \text{By associative property of addition.} \\ &= 5 + \tfrac{2}{3} \\ &= 5\tfrac{2}{3} \end{aligned}$$

In the subtraction problem below it will be convenient for us to rewrite rational numbers in fractional form. We have written the rational number $8\frac{1}{3}$ as $\frac{25}{3}$. Now let us write it in still another way. By using the associative law you may write $8\frac{1}{3} = 8 + \frac{1}{3} = (7 + 1) + \frac{1}{3} = 7 + (1 + \frac{1}{3}) = 7 + \frac{4}{3}$. Can you write out the reasoning, as is done above, for writing $7\frac{1}{4}$ as $6 + \frac{5}{4}$? For $12\frac{3}{8}$ as $11 + \frac{11}{8}$? For $13\frac{2}{3}$ as $12 + \frac{5}{3}$?

In the following problem:

$$8\frac{1}{3} - 2\frac{5}{6}$$

we find it convenient to write $8\frac{1}{3}$ as $7\frac{4}{3}$ since $\frac{5}{6}$ is greater than $\frac{1}{3}$. Therefore, $\frac{5}{6}$ is not greater than $\frac{4}{3}$.

$$\begin{aligned}
8\frac{1}{3} - 2\frac{5}{6} &= (8 + \frac{1}{3}) - (2 + \frac{5}{6}) \\
&= (7 + \frac{4}{3}) - (2 + \frac{5}{6}) \\
&= (7 + \frac{4 \cdot 2}{3 \cdot 2}) - (2 + \frac{5}{6}) \\
&= (7 + \frac{8}{6}) - (2 + \frac{5}{6}) \\
&= (7 - 2) + (\frac{8}{6} - \frac{5}{6}) \\
&= 5 + \frac{3}{6} \\
&= 5\frac{3}{6} \\
&= 5\frac{1}{2}
\end{aligned}$$

You may have written this

$$\begin{array}{rcccc}
8\frac{1}{3} & = & 7\frac{4}{3} & = & 7\frac{8}{6} \\
- \; 2\frac{5}{6} & = & 2\frac{5}{6} & = & 2\frac{5}{6} \\
\hline
 & & & & 5\frac{3}{6} = 5\frac{1}{2}.
\end{array}$$

Exercises 6-8c

1. Subtract. Express answers in simplest forms.

   (a) $\frac{15}{64} - \frac{7}{64}$

   (b) $\frac{24}{16} - \frac{9}{16}$

   (c) $\frac{8}{9} - \frac{2}{3}$

   (d) $3\frac{4}{5} - \frac{1}{3}$

   (e) $5\frac{9}{16} - 2\frac{5}{8}$

   (f) $\frac{14}{6} - \frac{11}{9}$

   (g) $(4\frac{1}{2} - 1\frac{1}{4}) - 2$

   (h) $16\frac{2}{3} - \frac{7}{3}$

   (i) $\frac{90}{100} - \frac{109}{1000}$

   (j) $(\frac{2}{3} \cdot \frac{5}{6}) - (\frac{2}{3} \cdot \frac{1}{6})$

   (k) $\frac{2}{3}(\frac{5}{6} - \frac{1}{6})$

   (l) Compare your results for j and k.

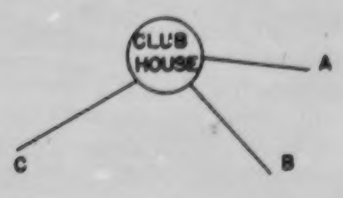
   (m) From your work in j, k, and l, do you think multiplication is distributive over subtraction?

2. John lives $\frac{5}{16}$ of a mile from school. Jane lives $\frac{3}{7}$ of a mile from school.

   (a) Estimate who lives the farthest from school.

   (b) Show that your estimate was correct.

*3. Vic, Mel, and Bob built a club house. Vic lives $\frac{2}{3}$ of a mile from the club house. Mel lives $\frac{5}{8}$ of a mile from the club house. Bob lives $\frac{7}{9}$ of a mile from the club house. In the diagram below, A represents the home of the boy nearest the club house, B, the next and C, the farthest. Which boy lives at A? at C? at B? How much farther is B from the club house than A? How much farther is C than A?



4. The length of a field is $30\frac{1}{2}$ rods. The width of the same field is $15\frac{3}{4}$ rods. Find the difference between the length and width of the field.

5. Jake ran the 100 yd. dash in $11\frac{1}{5}$ seconds. Bruce ran it in $10\frac{4}{5}$ seconds. How many more seconds did it take Jake to run the 100 yds. than it did Bruce?

6. Mary is making a chintz apron that requires $1\frac{5}{8}$ yds. of material. She has $4\frac{3}{4}$ yds. of chintz. How much material will she have left?

7. Jim left the house at 5:15 P.M. It took him $\frac{1}{4}$ hour to pick up the papers for delivery, $1\frac{1}{4}$ hours to deliver them, and $\frac{1}{4}$ hour to return home. He ate supper and was at the movies by 7:45 P.M. How much time did it take him to eat and get to the movies?

8. Mr. Twiggs changed the price of potatoes in his store from $4\frac{1}{2}$¢ a pound to 3 pounds for 14¢.

   (a) Did he raise or lower the price?

   (b) How much was the increase or decrease per pound?

9. (a) Is the difference of $17\frac{1}{4}$ and $6\frac{3}{4}$ more or less than 11?

   (b) How can you tell without working out the example?

10. Is the number represented by a fraction changed if you subtract the same number from both numerator and denominator? Give an example.

*11. Fill in the blanks in the magic square below so that the sum of the numbers in each row is the same. Then find the sum of the numbers of each column and both diagonals.

$\frac{2}{3}$	$\frac{1}{12}$	$\frac{1}{2}$
	$\frac{5}{12}$	$\frac{7}{12}$
$\frac{1}{3}$		$\frac{1}{6}$

6-9. Ratios Expressed by Rational Numbers

In your use of numbers, you probably noticed that you can compare two numbers by subtraction or by division. Of the two numbers, 6 and 2, we can say that the first number is 4 more than the second, or that the first number is three times the second. We can also say that the ratio of the first number to the second is 3 to 1 or 3. In comparing the numbers 9 and 2, we sometimes say that the ratio of these numbers is 9 to 2 or $\frac{9}{2}$. Nine is four and one-half times the number 2.

Definition: The ratio of a number c to a number d, $d \neq 0$, is $\frac{c}{d}$.

If c and d are counting numbers, the ratio $\frac{c}{d}$ is a rational number.

In a class there are 36 pupils of whom 16 are girls. The ratio of the number of girls in the class to the number of class members is $\frac{16}{36}$ or $\frac{4}{9}$.

Thus $\frac{4}{9}$ of the class members are girls: $36 \cdot (\frac{4}{9}) = 16$. The ratio of the number of pupils in the class to the number of girls in the class is $\frac{36}{16}$ or $\frac{9}{4}$. There are two and one-fourth times as many pupils as girls in the class. Notice that $\frac{9}{4} = 2 + \frac{1}{4}$.

If a car travels 258 miles in 6 hours, the ratio of the number of miles traveled to the number of hours of travel is $\frac{258}{6}$ or $\frac{43}{1}$ or 43. This ratio, 43, is usually called the rate that the car is traveling and is often expressed in terms of miles per hour. In examples of motion, like that of a moving car, you may have used a formula

$$d = rt,$$

where d represents number of units in the distance traveled; r, the rate of travel; t, the time of travel. If $d = 258$ and $t = 6$, then

$$6r = 258,$$
$$r = 43;$$

r is the ratio of 258 to 6 or the rational number $\frac{258}{6}$. Other names for $\frac{258}{6}$ are $\frac{43}{1}$ and 43.

Ratio is very useful in many applications of mathematics. In the remainder of the work, this year as well as next, we will see many other examples of ratios. Some of the applications of ratio are illustrated in the exercises.

Exercises 6-9

1. Express the ratios of the following as rational numbers in simplest form:
   - (a) 3 feet to 10 feet.
   - (b) A test grade of 75 to a test grade of 90.
   - (c) 56 pounds to 12 pounds.
   - (d) $\frac{9}{2}$ to $\frac{7}{2}$.
   - (e) $\frac{5}{4}$ inches to 2 inches.
   - (f) 90 cents to 75 cents.
   - (g) 15,000 people to 25,400 people.
   - (h) 90 degrees to 55 degrees.
2. A plane flies 2600 miles in 5 hours.
   - (a) At what rate does the plane fly per hour?
   - (b) What is the ratio of the number of miles traveled to the number of hours of flying time?
3. The ratio of 5 inches to one yard can be said to be $\frac{5}{1}$, or if we change yards to inches, the ratio is $\frac{5}{36}$. In using ratio we must be sure what units we are using. What advantage may there be in calling the ratio of 5 inches to one yard $\frac{5}{36}$?
4. The scale on a map is 1 inch to 20 miles.
   - (a) Express this scale as a ratio.
   - (b) On the map how many miles are represented by a segment of length $4\frac{1}{4}$ inches?

5. Centerville is a small city with city limits forming the sides of a rectangle, 3 miles in length on the longer side, and 2 miles in length on the shorter side. Using a scale of 1 inch for $\frac{1}{8}$ of a mile, how long and how wide will the map of the city have to be?

6. A map of the United States is drawn on a scale of 10 inches to 3000 miles.
   (a) Express this scale as a ratio.
   (b) On this map how many miles will be represented by one foot?
   (c) The distance from Washington to Chicago is approximately 750 miles. How far apart will these cities be on the map?

7. In a scale drawing of the floor plan of a school building 1 inch represents 6 feet.
   (a) Express this scale as a ratio.
   (b) On the scale drawing how many feet are represented by a segment of length of 20 inches?
   (c) In (b) what is the ratio of the length of the segment on the drawing to the number of feet represented by this segment?

8.

x	1	2	3	4	5	6
3x	3	6	9	12	15	18

Find the ratio of each number in the second line of the table to the corresponding number in the first line of the table.

9.

2	4	5	8	10		
5	10	$12\frac{1}{2}$	20		30	40

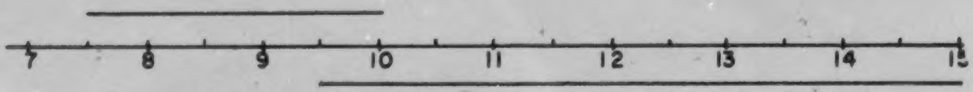
Assume the ratio of each pair of corresponding numbers (numbers in the same vertical column) in the table is the same. What are the missing numbers?

10. On a thermometer an inch of mercury represents a temperature of 40 degrees. (a) How many inches of mercury will represent a temperature of 60 degrees, assuming that the temperature scale starts at 0 degrees? (b) A temperature of 67 degrees?

11. A quart of canned orange juice sells for 54 cents and a gallon of the same juice sells for $1.98. Find the ratio of selling price in cents to the number of pints, in both cases.

12. The cost of a bicycle on January 1, 1961 is $\frac{11}{10}$ of the cost of the same bicycle on January 1, 1959. (a) What is the ratio of the cost on the latter date to the cost on January 1, 1961? (b) If the bicycle cost $66 on January 1, 1961, what was its cost two years ago?

*13.



The length of the bar above the scale is 2 inches. Find, without measuring, the length of the bar below the scale.

*14. A man is 5 feet 7 inches tall and his son is 6 feet 2 inches tall. (a) What is the ratio of the height of the father to the height of the son? (b) The son's height is how many times the father's height? (c) The father's height is how many times the son's?

6-10. Decimal Notation

In your everyday experiences you probably have used decimal notation for rational numbers. For example, we speak of a quarter to mean 25 cents or one-fourth (one quarter) of a dollar. We often write 0.25 as a name for $\frac{25}{100}$. Since $\frac{25}{100}$ is a name for one fourth, we see that 0.25, $\frac{25}{100}$, and $\frac{1}{4}$ are different names for the same rational number. As ratios, these number names in this example express the ratio of the number of cents in a quarter to the number of cents in a dollar.

Names for the rational number n, for which $10n = 3$ is a true statement, are $\frac{3}{10}$, 0.3, $\frac{30}{100}$, and 0.30. You will notice that the decimal notation for $\frac{3}{10}$ is 0.3. In printed symbols, or names for numbers between 0 and 1, a zero is usually used in the first place to the left of the decimal point so that the decimal point will not be overlooked. Also the 0 in the one's place emphasizes that there are no one's in the number or that the number is between 0 and 1.

Decimal notation for $\frac{3}{1000}$ is 0.003.

The numeral 1.87 is read one and eighty-seven hundredths. In money we would speak of $1.87. The decimal form, 1.87, is a name for $1 + \frac{87}{100}$, which in turn is a name for the rational number $\frac{187}{100}$.

Some "decimal" names we often use for rational numbers are:

$$\begin{array}{lll} \frac{1}{4} = \frac{25}{100} = 0.25 & \frac{1}{2} = \frac{50}{100} = 0.50 \text{ or } & \frac{1}{2} = \frac{5}{10} = 0.5 \\ \frac{3}{4} = \frac{75}{100} = 0.75 & \frac{3}{10} = 0.3 & \frac{1}{8} = \frac{125}{1000} = 0.125 \end{array}$$

Suppose we wanted to write $\frac{7}{8}$ in decimal notation. We know that $\frac{7}{8} = 7 \div 8$, so let us divide.

$$\begin{array}{r} 0.875 \\ 8\overline{)\,7.000} \end{array}$$

We see that $\frac{7}{8}$ may be written as 0.875. We can check this result by testing to see if $\frac{7}{8} = 0.875$ (or $\frac{875}{1000}$) is a true statement. We see that

$$\frac{875}{1000} = \frac{5 \cdot 7 \cdot 5 \cdot 5}{5 \cdot 5 \cdot 5 \cdot 2 \cdot 2 \cdot 2} = \frac{5}{5} \cdot \frac{5}{5} \cdot \frac{5}{5} \cdot \frac{7}{2 \cdot 2 \cdot 2} = \frac{7}{8}.$$

We might observe that, when using decimals for names of some rational numbers, the name of the number can be obtained by division.
For example,

$$\begin{array}{r} 0.75 \\ 4\overline{)3.00} \end{array} \qquad \frac{3}{4} = 0.75$$

$$\begin{array}{r} 0.3125 \\ 16\overline{)5.0000} \\ \underline{4\,8} \\ 20 \\ \underline{16} \\ 40 \\ \underline{32} \\ 80 \\ \underline{80} \end{array} \qquad \frac{5}{16} = 0.3125$$

0.3125 is read
three thousand one hundred
twenty-five ten thousandths.

$$0.3125 = 0(1) + 3(\frac{1}{10}) + 1(\frac{1}{100}) + 2(\frac{1}{1000}) + 5(\frac{1}{10,000}).$$

In addition of rational numbers we make use of the distributive property.
For example,

$$\begin{aligned} \frac{32}{100} + \frac{55}{100} &= 32(\frac{1}{100}) + 55(\frac{1}{100}) \\ &= (\frac{1}{100})(32 + 55) = (\frac{1}{100})(87) = \frac{87}{100}. \end{aligned}$$

If we wish to use decimal notation, we can replace $\frac{1}{100}$ by 0.01, $\frac{32}{100}$ by 0.32, and $\frac{55}{100}$ by 0.55,

$$\begin{aligned} 0.32 + 0.55 &= 32(0.01) + 55(0.01) \\ &= (0.01)(32 + 55) = (0.01)(87) = 0.87. \end{aligned}$$

If you are not sure of the last product, (0.01)(87), you may recall that

$$(0.01)(87) = (\frac{1}{100})(87) = \frac{87}{100} \quad \text{or} \quad 0.87.$$

This, and other examples, may convince you that decimal notation makes some work seem easier to do. If we wrote this addition problem in a column, we would need to write only

$$\begin{array}{r} 0.32 \\ +\ 0.55 \\ \hline 0.87. \end{array}$$

In decimal notation, we add tenths to tenths and hundredths to hundredths, using the distributive property.

Let us consider another example,

$$\frac{3}{4} + \frac{1}{2} = \frac{3}{4} + \frac{2}{4} = \frac{1}{4}(3 + 2) = \frac{1}{4}(5) = \frac{5}{4}$$

or in decimal notation

$$0.75 + 0.50 = 0.01(75 + 50) = 0.01(125) = 1.25.$$

In columnar form, we would have

$$\begin{array}{r} 0.75 \\ +\ 0.50 \\ \hline 1.25. \end{array}$$

In this example, we have used

$$\frac{3}{4} = \frac{75}{100} = 0.75, \qquad \frac{1}{2} = \frac{50}{100} = 0.50,$$

$$0.75 = (0.01)75, \qquad 0.50 = (0.01)50,$$

and $(0.01)125 = 1.25.$

In order to see why $(0.01)125 = 1.25$, replace 0.01 by $\frac{1}{100}$ to obtain

$$(0.01)125 = (\frac{1}{100})125 = \frac{125}{100} = 1 + \frac{25}{100} = 1.25.$$

Exercises 6-10

1. Express the rational numbers in decimal notation:

   (a) $\frac{1}{2}$

   (b) $\frac{1}{4}$

   (c) $\frac{3}{8}$

   (d) $\frac{7}{10}$

   (e) $\frac{34}{100}$

   (f) $\frac{5}{4}$

   (g) $\frac{11}{4}$

   (h) $\frac{8}{5}$

   (i) $\frac{5}{8}$

   (j) $\frac{1}{16}$

   (k) $\frac{15}{16}$

   (l) $\frac{60}{16}$

   In the last three parts use four decimal places.

2. Write the decimals as rational numbers in simplest form:

   (a) 0.75

   (b) 1.75

   (c) 0.6

   (d) 5.6

   (e) 0.36

   (f) 2.36

   (g) 0.140

   (h) 0.012

   (i) 0.825

   (j) 0.875

   (k) 1.506

   (l) 2.008

3. Write the decimals in Problem 2 in words.

4. Add: $10.57, $4.17, $5.92, $21.72, $3.06, $5.09

5. Add the rational numbers in fractional form and then change each to decimal notation and add: $\frac{5}{4}$, $\frac{1}{2}$, $\frac{3}{8}$, $\frac{5}{2}$, $\frac{5}{8}$

6. Add the numbers: 1.4, 3.75, 10.06, 3.7, .096, 9.99.

7. The lengths of several pieces of lumber have been measured to the nearest 0.1 of a foot. The lengths are:

   6.3, 8.2, 9.7, 1.6, 2.8, 3.9

   (a) Find the sum of these lengths.

   (b) Find the difference in length between the shortest and longest piece.

8. Find the products, changing the decimals to fractions if that helps you:

   (a) 4(0.01)

   (b) 25(0.1)

   (c) 356(0.001)

   (d) 240(0.01)

   (e) 1492(0.01)

   (f) 673(0.1)

   (g) (0.1)(0.01)

   *(h) (2.3)(1.2)

*9. Write in decimal notation correct to the nearest .001:

(a) $\frac{2}{3}$

(b) $\frac{5}{6}$

(c) $\frac{4}{9}$

(d) $\frac{1}{11}$

(e) $\frac{1}{9}$

(f) $2\frac{3}{8}$

(g) $\frac{41}{3}$

(h) $\frac{55}{7}$

(i) $\frac{1000}{6}$

*(j) $\frac{2134}{21}$

10. Use the distributive property to show why the sum of

3.25 and 6.71 is 9.96.

6-11. Ordering

We have had methods of finding whether two fractions represent the same number. It is often important to know which of two given unequal fractions represents the larger number. If, in one store you can get three apples for a dime, and in another two apples for a dime, it is easy to see that apples are cheaper in the first store since you get more for a dime there. This can also be seen by noticing that in the first store you get $\frac{3}{10}$ of an apple for a cent, and at the second you get only $\frac{2}{10}$ of an apple for a cent. We know that $\frac{3}{10}$ is greater than $\frac{2}{10}$, which can be written

$$\frac{3}{10} > \frac{2}{10} .$$

On the number line

0
$\frac{1}{10}$
$\frac{2}{10}$
$\frac{3}{10}$
$\frac{4}{10}$
$\frac{10}{10}$

$\frac{3}{10}$ occurs to the right of $\frac{2}{10}$.

The problem would be harder, if in the first store, you could get 3 apples for five cents and in the second store 8 apples for 13 cents. Then we would have to answer the question:

$$\text{Is } \frac{3}{5} > \frac{8}{13} \text{ ?}$$

It would require careful drawing to answer this by use of the number line. There are at least two better methods of finding the answer.

Method I. Convert both fractions into decimal form and compare the results. Then $\frac{3}{5} = 0.60$ and $\frac{8}{13} = 0.61\ldots$ as may be seen from the following division:

$$\begin{array}{r} 0.61 \\ 13\overline{)\,8.00} \\ \underline{7\,8} \\ 20 \\ \underline{13} \\ 7 \end{array}$$

It is only necessary to carry the division to two places to see that the decimal values of the two fractions are different.

Method II. Here we use the method of example 3 of section 5; that is, we find two fractions with equal denominators which represent the given numbers. Since the denominators are 5 and 13, the smallest denominator which we can use for both fractions will be the least common multiple of 5 and 13, which is 65. Then

$$\frac{3}{5} = \frac{3 \cdot 13}{5 \cdot 13} = \frac{39}{65} \quad \text{and} \quad \frac{8}{13} = \frac{8 \cdot 5}{13 \cdot 5} = \frac{40}{65} .$$

Since $\frac{40}{65} > \frac{39}{65}$ it is also true that $\frac{8}{13} > \frac{3}{5}$.

Either method shows that if we divide a number line into 65 equal divisions, the point representing $\frac{8}{13} = \frac{40}{65} = .615 \ldots$ lies just to the right of the point representing $\frac{3}{5} = \frac{39}{65} = .6000$.

A fraction in which the numerator is greater than the denominator is often called an improper fraction. The number which such a fraction represents must be greater than 1. To see why this is so, consider the fraction $\frac{13}{11}$. This is greater than $\frac{11}{11}$ which is equal to 1. Thus $\frac{13}{11}$ is greater than 1. Similarly $\frac{3568}{3452}$ is greater than 1 since it is greater than $\frac{3452}{3452}$ which is equal to 1.

Exercises 6-11

1. For each of the following pairs of fractions, tell which represents the larger number:

   (a) $\frac{7}{12}$ and $\frac{2}{3}$ (b) $\frac{4}{5}$ and $\frac{13}{16}$ (c) $\frac{13}{15}$ and $\frac{13}{7}$

2. Which is cheaper, five oranges for eleven cents or four oranges for nine cents?

3. Brand A of pineapple juice comes in cans of 15 ounces each and brand B in cans of 8 ounces each. If brand A sells for 23 cents a can and brand B for 12 cents a can, which is cheaper?

4. If the numerator of a fraction is greater than twice its denominator, show that the number which the fraction represents is greater than 2.

*5. If two fractions have equal denominators and if the numerator of the first is greater than the numerator of the second, then the number represented by the first is greater than the number represented by the second. Suppose two fractions have equal numerators; how can you tell by comparing the denominators which fraction represents the larger number? (Try a few pairs of fractions first to see how it goes.)

Chapter 7

MEASUREMENT

7-1. Counting and Measuring

Many of the questions people ask in their work or in everyday living begin with "How much?" or "How fast?" We ask, "How many people went to the ball game?" or "How much meat shall I buy?" or "How fast can a jet travel?" The answers we want for these questions are all alike in one respect: they all involve numbers. But the answers to some of them are found by counting, while answers to others are found by measuring.

When we ask, "How many people went to the ball game?" we find the answer by counting. Each person is a separate, whole object. There are no fractions of people at a ball game. They are not all exactly alike, nor are they all the same size, but each is a person. When we count them we use only the counting numbers, and our answer is a counting number. There are 78 or 79 people: it is impossible that there should be $78\frac{1}{2}$ or $79\frac{1}{4}$ people.

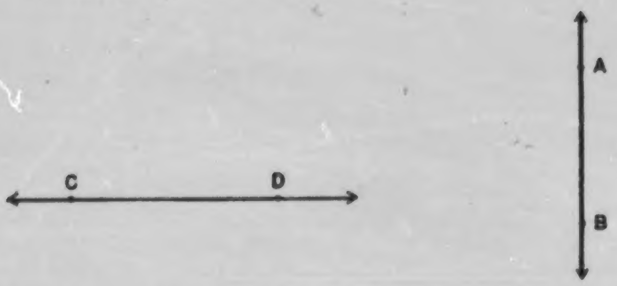
When we ask, "How long is this rope?" we cannot find the answer just by counting, because the rope is one continuous thing; it is not made up of separate parts which we can count. We must find how long it is by measuring. Then the answer is a number of units. We may say it is 45 inches long, or $3\frac{3}{4}$ feet long, or $1\frac{1}{4}$ yards long. We use a rational number to answer the question, but often not a counting number. To answer the question, "How fast can a jet travel?" we must use two units. We may say, "It can travel 1200 miles per hour," using a unit of distance and a unit of time.

In this chapter we are going to study ways to measure things that are continuous. When we ask the question, "How many?" we are thinking of a set of objects and wish to have some measure of how many there are. We sometimes call this a discrete set. When we ask such questions as, "How much?" "How long?" "How fast?" etc., we are seeking to describe something thought of as all in one piece, without any breaks. When we think of something this way, we call it a continuous set. Sets of people, houses, animals, marbles or jacks, are discrete sets; we want to know "How many?" A rope, a wire, a road, or a flagpole are all thought of as being continuous since they are like line segments; we can count a number of line segments but we cannot count the number of all points on a line segment. Cloth, a blackboard, a football field, and a pasture may be **thought of as regions** enclosed by simple closed curves, and hence are continuous; we can count football fields but we do not usually count a football field.

Properties of Continuous Quantities

Continuous quantities which we wish to measure may have different natures. Some continuous quantities or sets are thought of as line segments. Some may be thought of as enclosed by simple closed curves, and some may be thought of as volumes or capacities. For example, your height or your dog's leash may be thought of as a line segment. Your gymnasium floor may be thought of as the region enclosed by a simple closed curve.

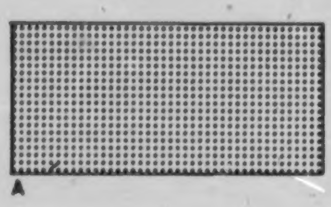
The sizes of line segments and of simple closed curves can be compared without actually measuring them. To see which of two boys is taller, the boys might stand back-to-back to compare the position of the tops of their heads. A similar method can be used to find out which of $\overline{AB}$ or $\overline{CD}$, shown below, is longer. The use of the symbol $\overline{AB}$ was taught in Chapter 4: it is a short way of writing "the segment AB."

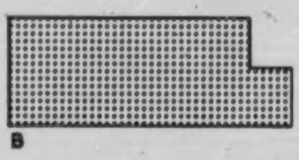


To compare these segments, lay the edge of a piece of paper along $\overline{CD}$ and mark points C and D on this paper so that they match the points in the picture. Place the edge of the paper along $\overline{AB}$ with point C on point A. Where does point D fall? If it is between A and B, $\overline{AB}$ is longer than $\overline{CD}$. If D falls on B, the segments are the same length. We say that segments of the same length are *congruent*. If B is between C and D, $\overline{CD}$ is longer than $\overline{AB}$.

We wish to consider the set of all points which either lie on a simple closed curve or are contained in the interior of the simple closed curve. Such a set of points will be called a *closed region*.

Look at the closed regions below. Copy them by tracing them on a sheet of thin paper.


A


B

To compare the size of the closed region A with the size of the closed region B, cut out a copy of figure B and place it on the closed region A. Figure B separates A into the part covered by B and the part not covered by B. We say that the *area* of A is larger than the *area* of B.

When we use methods like this to compare the sizes of line segments or of closed regions, we assume these properties of geometric continuous quantities:

Motion Property. A geometric figure may be moved without changing its size or shape.

Comparison Property. Two continuous geometric figures or sets, *of the same kind* may be compared to determine whether they have the same size, or which one is the larger.

Exercises 7-1

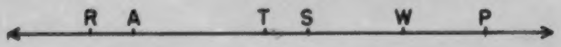
1. Indicate which of the following questions you would answer by counting and which by measuring.

   (a) How many people are in this class?
   (b) How far is it to the moon?
   (c) How big a crowd went to the picnic?
   (d) How cold is it today?
   (e) How old are you?
   (f) How much water is there in the swimming pool?

2. Tell which of the following are continuous quantities and which are discrete objects.

   (a) Your height.
   (b) Your weight.
   (c) The size of your family.
   (d) The length of this page.
   (e) The odd numbers between 0 and 10.
   (f) The amount of air in this room.

3. Use a strip of paper or string of the same length as $\overline{RT}$ located on the following line:



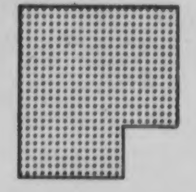
   Use $\overline{RT}$ to compare the following line segments. Indicate whether the segments named are congruent or which of the two is longer.

   (a) $\overline{RT}$ and $\overline{SP}$. (b) $\overline{RT}$ and $\overline{TW}$. (c) $\overline{RT}$ and $\overline{AS}$.
   (d) $\overline{RT}$ and $\overline{AT}$. (e) $\overline{RT}$ and $\overline{TP}$.

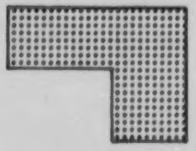
4. Select the closed region below which you think is the smallest. Write sentences comparing the size of the smallest closed region with the size of the closed regions of each of the other figures. Use one of the symbols, >, =, <, and the form shown here:

The closed region of figure _?_ < the closed region of figure _?_ .

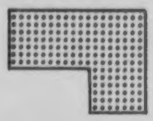
A



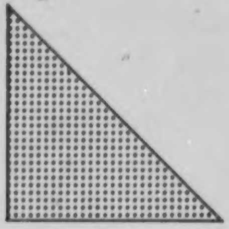
B



C

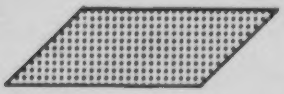


D

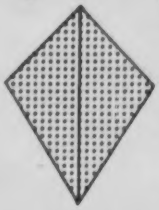


5. (a) Which of the four closed regions below do you think is the largest? List them in what you think is the order of their sizes, putting the largest one first.

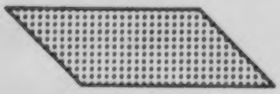
A



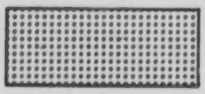
B



C



D



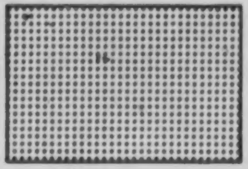
(b) Make three copies of Figure A, and use them to compare the closed region A with the other closed regions. (Cut up the copies of Figure A if necessary.)

(c) Was the list you made for Problem 5a correct?

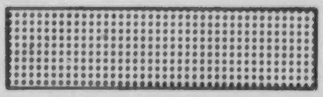
6. (a) Use the edge of a piece of paper to mark off the lengths of the four line segments in the closed curve in Figure E below. Then use the strip to compare the length of the closed curve in Figure E with the length of the closed curve in Figure F. Which curve has the greater length?

(b) Use a copy of Figure E to compare the sizes of the closed regions E and F. Which closed region has the larger area?

E



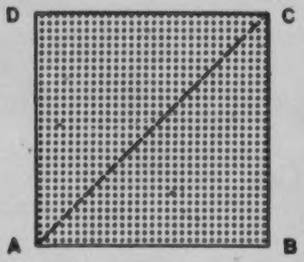
F



7. Draw a square and label the vertices ABCD. Cut out your square closed region and then cut along $\overline{AC}$. Put the two parts together in some other way so that they do not form a square, and trace around the boundary of the new figure.

(a) How does the new closed region compare in area with the closed region of square ABCD?

(b) How do the lengths of the closed curves compare?



In Problems 4-7 we have assumed two more properties of geometric continuous figures, or sets.

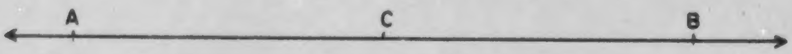
Matching Property. If two geometric continuous figures, or sets, are both made up of parts such that every part of one can be matched to a part of the same size in the other, then the two continuous figures, or sets have the same size.

Subdivision Property. A geometric continuous figure or set may be subdivided.

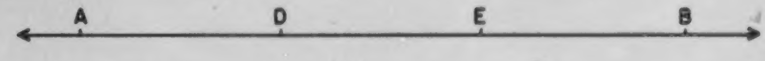
7-2. Subdivision and Measurement

The subdivision property of a continuous quantity is the basis for the process we call measuring. We will use some examples to help you understand this idea.

Each of the next three figures is a picture of a line containing the same segment, $\overline{AB}$. Point C subdivides segment $\overline{AB}$ into two segments $\overline{AC}$ and $\overline{CB}$. How do segments $\overline{AC}$ and $\overline{BC}$ compare in size? Are they congruent?

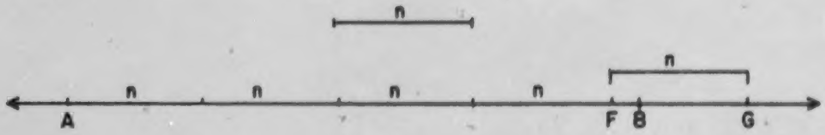


In the next sketch, $\overline{AB}$ is subdivided by two points D and E, so that the length of $\overline{AD}$ is $\frac{1}{3}$ the length of $\overline{AB}$ and the length of segment $\overline{EB}$ is also $\frac{1}{3}$ the length of $\overline{AB}$.



How does the length of $\overline{AE}$ compare with the length of $\overline{AB}$? Of $\overline{DB}$? Of $\overline{DE}$?

$\overline{AB}$ can be subdivided in other ways so as to compare the length of one line segment with the length of another line segment. Suppose we choose a segment of any length less than the length of $\overline{AB}$. We shall call the length of the segment "n".

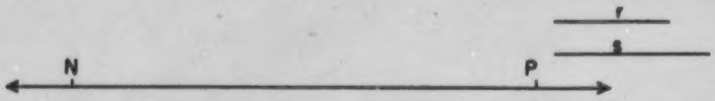


Use a strip of paper or a compass and mark off segments of length n on $\overline{AB}$. (Begin at point A.) In the figure above a segment of length n is marked off 4 times so that $\overline{AF}$ is as long as 4n. The symbol "4n" means "four times as long as n."

Point F is between A and B. If a fifth segment of length n is marked off, segment $\overline{AG}$ is as long as 5n, and B is between A and G. So $\overline{AB}$ is longer than 4n but shorter than 5n. Since $\overline{BG}$ is obviously longer than $\overline{FB}$, we say that $\overline{AB}$ is approximately equal to 4n. There is a symbol for the words "approximately equal to". It is a wavy equals sign like this, "$\approx$". We could use this symbol and state that the length of $\overline{AB} \approx 4n$.

In the previous example we compared the length of $\overline{AB}$ with the length of another segment whose length is called n. We did this by subdividing $\overline{AB}$ into congruent parts, each of which has the length n. We found that the length of $\overline{AB} \approx 4n$. This process is called measuring and the segment of length, n, is a *unit of measurement*. The number, 4, is the *measure* of AB when it is measured by unit n. ($\overline{AF}$ also has the measure 4.)

Make two measuring units, one congruent to segment r and the other congruent to segment s. Measure $\overline{NP}$ with unit r. How many r units in length is $\overline{NP}$?



Measure $\overline{NP}$ with the s unit. How many s units in length is $\overline{NP}$?

You should see that if you measure a segment with units of different size, the measure obtained does not tell you the size. You must also know the size of the unit of measurement that is used. The measure of $\overline{NP}$ is not the same when the unit r is used as when the unit s is used.

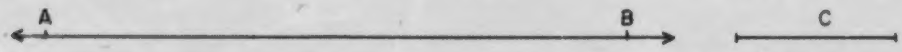
The length of a segment includes both the measure and the unit of measurement. In the example where we used n as a unit, the length of $\overline{AB} \approx 4n$. The symbol AB (with no bar above it) means the length of segment $\overline{AB}$. The length can be given by the statement, $AB \approx 4n$.

Notice how these words are used.

4 is the measure.
n is the unit of measurment.
4n is the length.

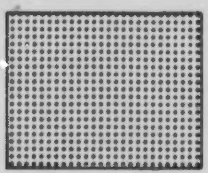
Exercises 7-2

1. Copy on paper segment $\overline{AB}$ and the line segment marked "c" below (or use a compass) to compare the length of $\overline{AB}$ with the length of segment c.

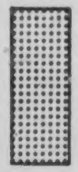


2. (a) Use the closed region B as a unit to compare the areas of the two closed regions. Copy Figures A and B. Cut out or trace Figure B and subdivide the closed region A into parts the same size as B.

A



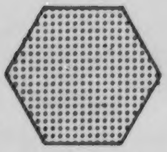
B



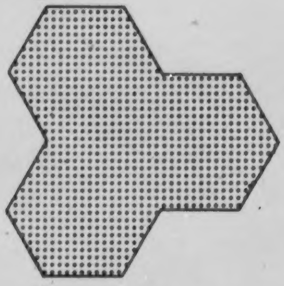
(b) The size of closed region A is approximately ______ times the size of B.

3. (a) Compare the length of curve D with the length of curve C.

C



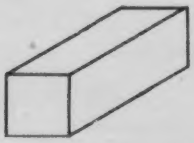
D



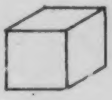
(b) Compare the size of the closed region D with the size of the closed region C.

4. Look at the box A in the figure. We could think of the solid A as subdivided into parts the size of box B, as shown in C. The solid box A is about _____ times as large as the solid B.

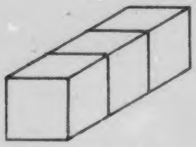
A



B



C



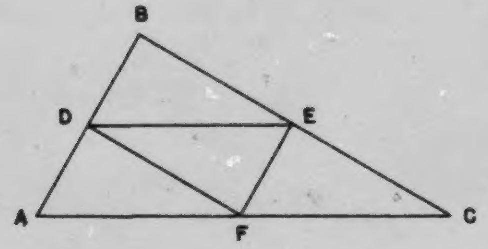
5. (a) What unit of measurement did you use in Problem 1?
   (b) What was the measure of $\overline{AB}$?

6. (a) What unit of measurement did you use in Problem 2?
   (b) What was the measure of the closed region A?

7. (a) In Problem 3a, suppose you use one segment of the curve C as a unit of measurement, and call the measure of this segment "t". Then what is the length of curve C?
   (b) Using the same unit of measurement "t", how long is curve D?
   (c) How does the length of curve D compare with the length of curve C?
   (d) In Problem 3b, what unit of measurement did you use? What is the size of the closed region D?

8. In Problem 4, what was used as the unit of measurement to measure the volume of Box A? How large is solid A?

9. Draw a triangle ABC. On each side of the triangle place a point which subdivides the side into two congruent segments. Name these points D, E, and F, as shown in the sketch. Draw segments $\overline{DE}$, $\overline{EF}$, and $\overline{DF}$.



   (a) Use a strip of paper to determine which segments are congruent. (You should find three sets, with three segments in each set.)
   (b) There are 11 simple closed curves in this figure. Some are triangles and some are four-sided figures. Name all the simple closed curves.

(c) Copy triangle EFC by tracing it, and compare the area of the closed region EFC with the areas of the other triangular closed regions.

*(d) What sets of closed regions are the same size? (One set should have four members and two sets should each have 3 members.)

7-3. Subdividing Units of Measurment

You have seen that the sizes of continuous quantities can be found by measuring. To do this, you must use a unit of measurement. The unit of measurement must have two characteristics:

1. It must be of the same nature as the thing to be measured--a line segment to measure a line segment, a closed region to measure a closed region, and so on.
2. It must be possible to move the unit, or to copy it accurately, so that it can be used to subdivide the thing that is measured.

Often people use anything suitable which is conveniently at hand for units of measurement. Things which you always have with you are your hands, fingers, arms, and feet. These parts of the body are often used as units to measure line segments.

Class Exercises 7-3

1. Use the length of the middle section of your little finger as a unit of measurement to measure
   (a) the length of your desk;
   (b) the length of this page.
2. Use the length of your foot to measure the length of your classroom.
3. Compare your answers to Problems 1 and 2 with the answers of your classmates. How do you explain any differences?

4. What object would make a convenient unit to measure the volume suggested by a desk drawer?

5. Can you think of some object that can be moved and that could be used as a unit of measurement for the surface of your desk? Could you use a sheet of paper from your notebook? Place several sheets side by side, and count the number of notebook-sheet units on the surface of your desk. What is the measure of the surface? How large is the surface?

6. Did you and all of your classmates use the same sized notebook-sheet unit? If not, what was true of the measure?

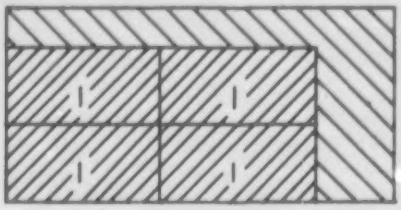
Cutting Units of Measurement

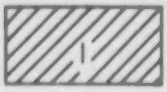
Probably your notebook-sheet units did not exactly cover the desk top a whole number of times. In such cases we often cover the uncovered surface with parts of units. Let us see how this is done.

You know what a rectangle is. It is a simple closed curve composed of four line segments, with intersecting line segments forming four "square corners." The four edges of a page in this book form a rectangle. Find some rectangles in your classroom. For each, point out the square corners and the interior of the rectangle.

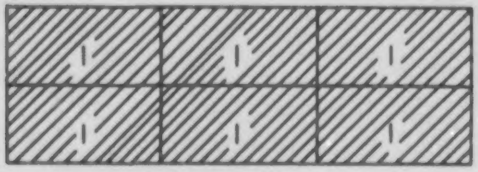
Suppose we wish to measure the large rectangular closed region in the next diagram using the small rectangular closed region as a unit.

Units may be laid side by side on the large rectangular closed region as shown to cover it as completely as possible.





It may happen that, as nearly as we can tell, the rectangular closed region can be covered "exactly" by some arrangement of the units, as shown in the figure.



The measure of the rectangular closed region is then obtained simply by counting the number of units. If we call the number of units in this region "A," we can write $A \approx 6$ to mean, "The size of this rectangular closed region is about 6 units." Notice that the symbol " $\approx$ " for "approximately equal" has been used, since it is not possible to lay on the units so that we can say the measure is exactly 6.

The size of a closed region is called its area. We say that the area of the rectangle above is 6 units. By this we mean that the size of the rectangular closed region is 6 units. Strictly speaking, a rectangle does not have area since it is a simple closed curve. We are likely to find that we can cover only a part of the rectangular closed region. Some of this region will stick out around the edges of the measuring units. The part left over will not be shaped so that we can fit any more whole units on it. In the figure below, the shaded part (/////) is not covered.

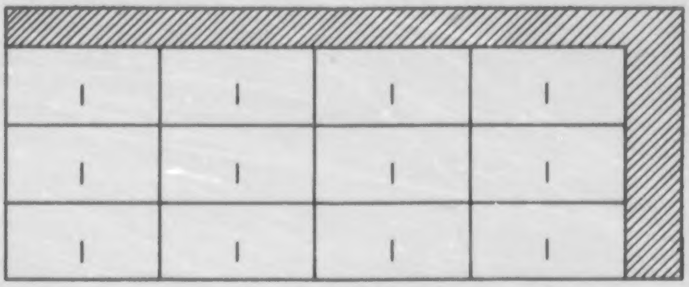


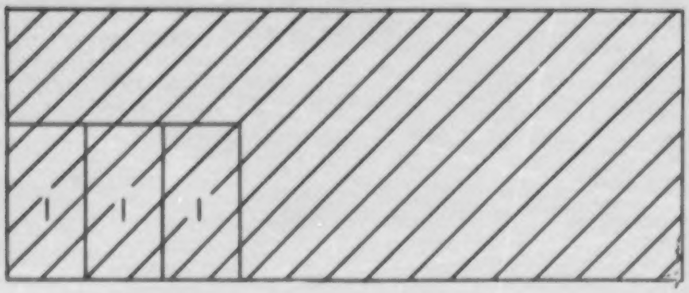
Figure 7-1

We can deal with this situation by cutting some of our units into smaller pieces. We can use these to cover the exposed portions. In doing this patchwork we must be careful to cut only one unit at a time. We must use all the pieces of one unit before starting to cut the next one. (Why must we use all of one unit before cutting another unit?)

Make several rectangular units of the size used here. Cut them, one at a time, to cover the exposed part of Figure 7-1. How many of these units did you use? Did you have a part of a unit left over which was not used? If so, does the left-over part seem to be more or less than half a unit?

If you have worked with reasonable care, you have probably used three full units and cut some from another. If the unused piece left over from this last unit is more than half a unit, you would write the area of the rectangle as approximately 15 units, since it seems to be as much as 15 but not as much as $15\frac{1}{2}$ units. If the remaining piece seems to be less than half a unit, you must have used more than half a unit, so you would write the area as approximately 16 units. It is not as much as 16 but is more than $15\frac{1}{2}$. Which of these answers did you find?

In the measuring just done, we have assumed that the result does not depend on how the rectangular units are placed on the large figure. For example, it is unlikely that all your classmates cut the units in the same way in piecing out the border. Yet we have assumed that the same answer should be obtained by everyone. The assumption that the area obtained does not depend on the way the units are placed on a figure can be tested. Measure the same rectangular closed region above with the same unit, but lay on the units so that the longer sides of the units are vertical, as shown on page 258, rather than horizontal.

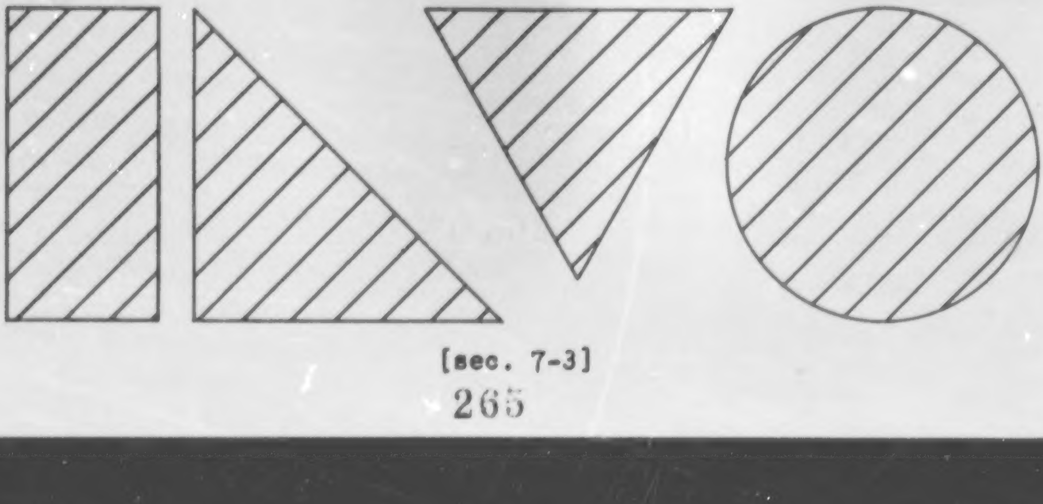


Do you find that the area of the rectangle is the same as before? Should it be? The assumption that a continuous quantity can be measured by covering it with units in any way you find convenient is very important. If we found different areas for our rectangle, then a single rectangle would seem to have two different sizes. To agree with our assumption that continuous quantities of the same kind can be compared, we accept the fact that the whole rectangle must have greater area than any part of it.

To measure this rectangular closed region we used as a unit of measurement a smaller rectangular closed region. We could equally well have chosen some other figure as a unit of area.

Exercises 7-3

1. Copy the figures below for use as units of area, and cut out several copies of each figure.



(a) Measure the area of a sheet of notebook paper, using each of these figures as a unit of measurement.

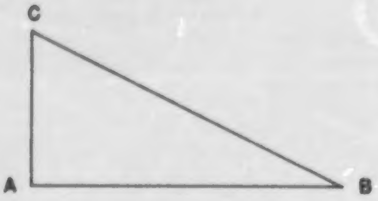
(b) Were any of the figures hard to use as a unit of measurement? If so, what was the reason?

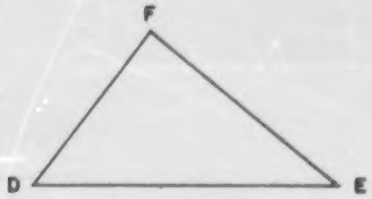
(c) Was the measure the same number for any two of the figures?

2. If you are a Boy Scout, you probably measure distances by "steps" or "paces." How long is your classroom when measured by your "pace"?

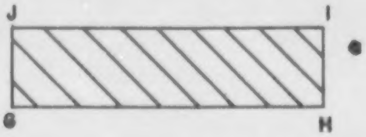
3. Could you measure the space occupied by a box by finding how many marbles of a certain size it will hold? Can you think of a differently shaped object which would be better to measure this space?

4. BRAINBUSTER. Look at the two triangles shown below. Angle CAB is a right angle, $\overline{AB}$ has the same length as $\overline{DE}$, and the triangles have heights of the same length.



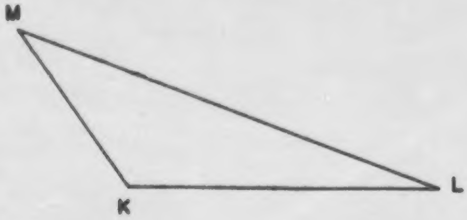


(a) Copy the triangles and cut out the triangular closed regions. Cut each closed region into pieces which can be assembled to form a rectangular closed region like the one below, in which $\overline{GH}$ has the same length as $\overline{AB}$.



(b) Cut one of the triangular closed regions into pieces which can be assembled to form the other triangular closed region.

(c) If you found this problem too easy, try it again when the second triangle looks like this.



7-4. Standard Units of Length

In Section 3 you used measuring units of various kinds. Some of these you made so that your unit was different from those of your classmates even when they were made in the same way. Other units that you used were the same for everyone in the class. When a unit is agreed upon and used by a large number of people, it is a standard unit. You can buy a ruler with the same size units as your classmates' rulers so easily that it may seem strange to you to learn that it took a long time and much hard work to see that making these units the same would simplify things. Primitive people have little need for a unit that is the same as everyone else's. If a caveman liked a bow that belonged to a friend of his, he could borrow it long enough to find a piece of wood that matched. What few possessions early man had could be exchanged without worrying much about the size. When civilization developed, and man began to have items of value, then the size was important. If one person has a diamond and another wants to give gold for it, the first person wants to be sure that the gold is pure and the weight honest. The story is told that King Hiero of Syracuse hired the Greek mathematician Archimedes to find a way to tell whether the goldsmith who had made his new gold crown had cheated him. Archimedes at first found the problem difficult but finally discovered a way to do this. The idea came to him while he was taking a bath and he was so excited that he jumped out and without waiting to dress, ran through the streets yelling "Eureka" which means "I found it." The goldsmith lost his head because a mathematician found a new way to measure.

This section deals with linear measures. "Linear" is an adjective and means "having the nature of a line." All measurement of line segments is called linear measurement. The small units of measure in the English system of measurement, which is the one we use in the United States, originally, were suggested by some part of the body. The yard is roughly the distance from the tip of the nose to the end of the fingers when the arm is held shoulder high. Of course if your nose itches and you twist a little, you change the size of your "yard" but for a rough estimate of how much cloth or rope you have, this is a pretty good measure. Horses are still measured by a unit called a hand. A hand was the width across the palm of the hand and is now the same as four inches. When boys play marbles, they use still another old measure. When you spread your thumb and little finger as far apart as possible, you have a unit called a span. When Noah built the ark it was supposed to be 300 cubits long, 50 cubits wide and 30 cubits high. The cubit was the distance from the elbow to the tip of the fingers, and it has varied from about 15 to 20 inches at different times in history. Sailors use a measure called a fathom. Originally they held a rope taut between their outstretched finger tips and tied knots at the two places where they held the rope. Now the fathom is 6 feet. When measuring longer distances, the Romans did the same thing that Boy Scouts do; they measured the number of steps it took to cover the distance. The Roman pace, however, was a double step. The Latin words for "a thousand paces" are "mille passus" and from the word for 1,000 comes our word "mile."

You have already discovered that body measures vary greatly on different people. This fact has led to many arguments throughout history. In early times, a tribe would decide on some standard and use it for that group. This worked well until someone in the group wanted to trade with someone outside of the group who had a standard of his own. Then the two tribes might fight over whose unit to use. As measures became more important, each country established a set of standards for all people who lived in that country. These units were established in many ways, but many of them still used the old body measures as the basis for their standard units.

King Edward I had a piece of iron made which was to be the official yard in England, but then he kept it locked up so that only the king and his close friends could use it. It is said that one of the English kings gave an order that his officers were to go to a certain church on a particular Sunday, take the first sixteen men who came out of the church and line them up so that the toe of one man just touched the heel of the man in front of him. This length, which was easily measurable with a rope, was divided into 16 equal parts and became the official "foot" in England at that time. In many countries similar actions were taken so that all of the people of the country (or in some cases it was a city) used the same measure. One village in France had a bar hanging in the market place; that bar was the official unit of length there.

This was all right until someone wanted to trade with somebody in another country. Disagreements were so common that finally a group of French scientists called a conference of representatives from many countries to meet with them and establish an international set of measures. This group developed the metric system which discarded the old body units and based all of their units on the distance from the North Pole to the equator. The meter is the basic unit of length of the metric system; it was planned to be one ten-millionth of the distance from the North Pole to the Equator. Recently an international congress of scientists defined the meter in relation to light waves. This system is used by all scientists in the world and is in common use in all countries except those in which English is the main language spoken.

The National Bureau of Standards in Washington has an accurate copy of the meter. This bar is made of platinum and iridium, a metal which changes very little in length under different atmospheric conditions. Congress has passed a law that tells what part of this bar shall be the official yard in the United States. This bar is the standard unit of length with which all standard units of length in this country are compared. The bar is considered so important that it is locked up very securely. If a manufacturer wants to make measuring instruments, his products must match the official standard.

Exercises 7-4a

1. Find out whether a loaf of bread is a standard unit. If it is, is there more than one standard "loaf"? If it is, what is the unit in the standard?
2. Are the tins for canned food made in standard sizes? What are these standards, if they exist?
3. List at least five items that are sold commonly and come in some kind of standard unit. Describe the standard unit for each item.

You have been using different kinds of units to measure line segments and closed regions of both the plane and space. Among others, you have used the length of the middle section of your finger, the length of your foot, a piece of notebook paper, and marbles. From the work that you have done with these, you should have developed the following ideas:

1. The size of a measuring unit was made up by somebody.
2. When a great many people decided to use the same unit, the unit became a standard unit.

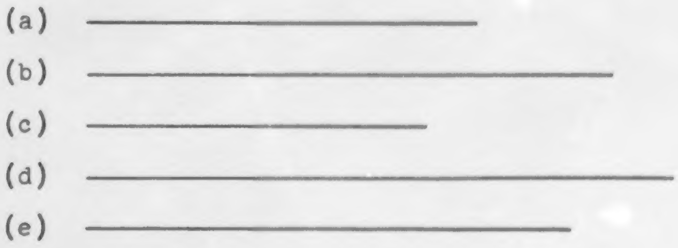
The inch was developed from the length of a section of a finger and the foot ruler was developed from the length of someone's foot. You also found out that your "inch" differs from your classmates' "inch," and the same thing is true of your "foot." The unit that we call the inch could be larger or smaller and still be a "unit" of measurement. By agreement, everyone who uses the English system of measures, uses a unit the length of your ruler as 1 "foot" and one-twelfth of this length as the unit 1 "inch."

Exercises 7-4b

1. Use a piece of tagboard or cardboard with a straight edge and make a 6-inch ruler marked at $\frac{1}{2}$ inch intervals.
2. Make two more 6-inch rulers, one marked with $\frac{1}{4}$ inch intervals and the other with $\frac{1}{8}$ inch intervals.

Use the symbol $\approx$ in your answers to problems 3 through 7.

3. Measure each of these line segments to the nearest half inch using the ruler that you made in Problem 1.



4. Measure each of the above line segments to the nearest $\frac{1}{4}$ inch, using your $\frac{1}{4}$ inch ruler.
5. Measure each of the above line segments to the nearest $\frac{1}{8}$ inch, using your $\frac{1}{8}$ inch ruler.

6. Measure each of the above line segments using the middle section of your third finger.
7. Which measuring unit gave you the most satisfactory results? Why?

The Ruler and the Number Line

In previous chapters you have worked with a number line. Measuring segments can be considered the same as comparing the segment that is to be measured with a number line. Actually, the ruler you use is part of a number line on which the unit is an inch. The part of the number line shown on the ruler starts at zero and goes to 12. A yardstick is a longer part of a number line that goes from 0 to 36. When you first worked with a number line, you put in only the counting numbers. When you learned what rational numbers are, you included them on the number line. Rulers that small children use are often marked only with inches and $\frac{1}{2}$ inches, like the first cardboard unit that you made. As children grow older they can understand rulers with smaller divisions. Most of the rulers that you use have each inch divided into 16 equal parts. When you measure a line segment, you compare the length of the segment with the number line marked on the ruler.

Class Exercises 7-4

Let's stop and take a careful look at the construction of the ruler you probably have.

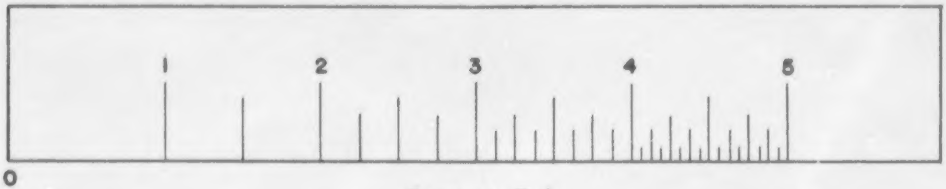


Figure 7-4

This ruler is divided into 12 equal units that everyone knows as inches. Part of it is shown in Figure 7-4. The space at each end of the figure is an inch long.

1. Why is it necessary to divide the units?
2. How is the inch between 1 and 2 divided?
3. How are the other inches divided?
4. When divisions smaller than those shown are needed, how are they obtained?
5. How many divisions are there in the third inch? The fourth inch? The fifth inch?
6. By looking at the ruler, how can you tell which divisions were made first?
7. Do these longer line segments help you in any way?
8. How many divisions is it possible to place between any two of the smallest divisions on your ruler?

If you have difficulty in using a ruler, study the above questions carefully and examine either the figure or your own ruler as you answer the questions.

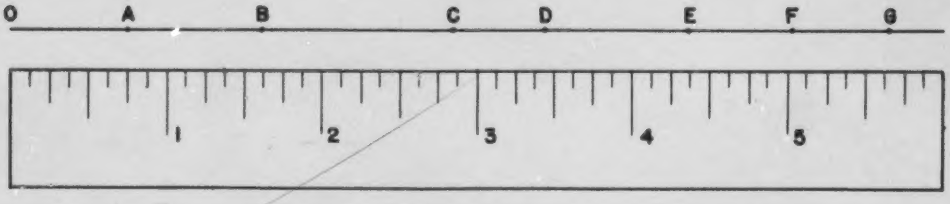
A ruler, or any other instrument, is of no help to you unless you can use it. A ruler has two uses: (1) A ruler can be used to draw a representation of a straight line. (2) A ruler is used to measure distances in terms of a standard unit, such as the inch.

One way of using the ruler to measure a distance is to place the zero point of the ruler at one endpoint of the segment, lay the ruler carefully along the line segment, and read the number of the point on the ruler that matches the other endpoint of the line. Unless your ruler has a small space to the left of the zero point, this is not the best way to use your ruler because most rulers wear at the corner, cutting off part of the first inch. A better measurement is made by matching the left end of the line segment with some other mark on the ruler, measuring the distance to the right end of the segment and subtracting the distance to the left of the first mark that you used. What is a convenient point to use at the left end of the segment?

Exercises 7-4c

1. If the sixth inch on Figure 7-4 were filled in, how many divisions would there be?

2.



What point on the ruler is below each of the points, A through G, on the line segment?

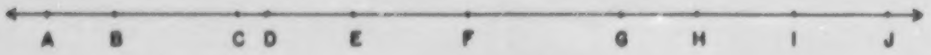
3. How long is:

   (a) $\overline{AB}$? (b) $\overline{AC}$? (c) $\overline{CD}$? (d) $\overline{DF}$? (e) $\overline{FG}$?

4. (a) Draw a segment 5 inches long and divide it into sections each $\frac{5}{8}$ inch long.
   (b) Divide 5 by $\frac{5}{8}$.
   (c) Is there any relation between parts (a) and (b) of this problem?

5. (a) Draw a line segment across your paper and mark these segments on it so that the left end of one starts where the right end of the preceding segment falls:
   $2\frac{1}{2}$" ; $1\frac{1}{4}$" ; $\frac{5}{8}$" ; $\frac{5}{16}$" .
   (b) What is the total length of these segments? Read this answer on your ruler.
   (c) Check your answer by addition.

6. With your ruler measure the segments indicated on the line below:



(a) $\overline{AB}$

(b) $\overline{BE}$

(c) $\overline{AJ}$

(d) $\overline{CG}$

(e) $\overline{EF}$

(f) $\overline{EJ}$

(g) $\overline{GH}$

(h) $\overline{IJ}$

7. (a) On the line in Problem 6, measure each of the following, if you have not already done so: $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, [illegible]. $\overline{EF}$, $\overline{FG}$, $\overline{GH}$, $\overline{HI}$, and $\overline{IJ}$.

   (b) Add all of these measures and check with the answer in 6 (c) to see if they agree.

8. Write the fractions $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{8}$, and $\frac{1}{16}$ as common fractions in base two.

*9. Is there any relation between the division on a ruler and base 2?

There are many standard units of length, such as the inch, foot, yard, or mile. Just as there are many names for the same rational number like $\frac{2}{3}$, $\frac{4}{6}$, and $\frac{10}{15}$, so there are many names for the same length, such as 2 feet, 24 inches, and $\frac{2}{3}$ yard. It is important to be able to change the name of a measurement from one form to another. To change these names, it is necessary to know the relationships among the units.

It is also necessary to consider the meaning of the symbol " = ". Up to this time this symbol has been used to show different names for the same length. When working with measurements, this meaning for the equal sign is used. With this in mind, we write 2 feet = 24 inches.

What is meant by "2 inches + 3 inches."? We know how to add numbers, but "adding lengths" is something different. Suppose we have two line segments of lengths 2 inches and 3 inches. Their measures, in inches, are 2 and 3. Mark these line segments on a line so that they have just one point in common. We get a segment whose measure, in inches, is 5 and whose length is 5 inches. Actually, the addition is performed on the numbers 2 and 3. This is what we will mean by

2 inches + 3 inches = 5 inches.

The symbol + as used above means something different from what it does when we write $2 + 3 = 5$.

In similar fashion we give meaning to 5 inches - 3 inches = 2 inches. The operation is actually performed with the numbers.

We know that 2×3 means $3 + 3$. Similarly, by 2(3 inches) we will mean 3 inches + 3 inches.

As an example, let us consider problems involving changing inches to feet and feet to inches. First, we will work with the problem of changing 60 inches to feet. We know the basic fact that 12 inches = 1 ft. This means that 1 in. = $\frac{1}{12}$ ft., that is, 1 in. and $\frac{1}{12}$ ft. are different names for the same length. We can write the problem as a number sentence, thus: 60 in. = _?_ ft. From the meaning of measurement, 60 inches is $60 \cdot 1$ in. With this in mind, the number sentence can be rewritten as $60 \cdot 1$ in. = _?_ ft.

Since 1 in. and $\frac{1}{12}$ ft. are different names for the same length, we can replace 1 inch by $\frac{1}{12}$ ft. Hence $60 \cdot \frac{1}{12}$ ft. = _?_ ft.

Now the arithmetic on the left side of the number sentence can be done and we see that 60 in. = 60(1 in.) = 60($\frac{1}{12}$ ft.) = $\frac{60}{12}$ ft. = 5 ft.

The same thinking is used when 50 ft. is changed to inches. The basic fact is the same, so that 50 ft. = 50(1 ft.) = 50(12 in.) = 600 in.

Common sense also helps in checking to see whether answers are reasonable. If the units used in measuring are small, it will take more of them to show a given measurement than when the units are large. Always check your answers to see if they are reasonable.

Most of the basic relationships needed to convert (or change the name of) units are familiar to you. They are given in the tables at the end of this chapter. The table of common linear measurements is also given here:

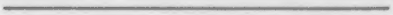
12 in. = 1 ft.
36 in. = 1 yd.
3 ft. = 1 yd.
5,280 ft. = 1 mi.

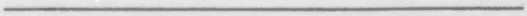
Exercises 7-4d

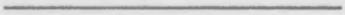
Where measurement is called for, use the symbol $\approx$.

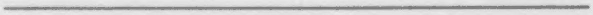
1. Walk naturally for ten steps, measure the distance from the first toe mark to the last toe mark and divide by ten to find the length of your pace. Express this length both in inches and in feet.

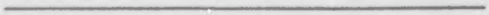
2. Use a ruler marked in 16ths of an inch to measure each segment below:

(a) 

(b) 

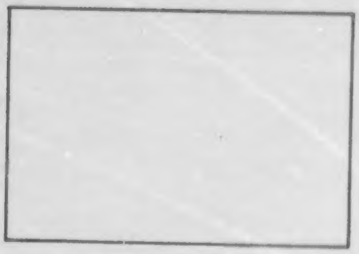
(c) 

(d) 

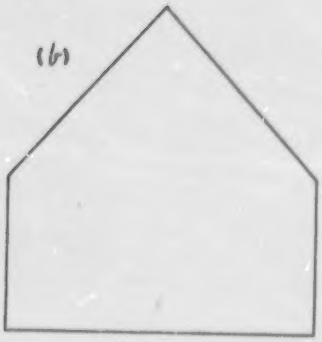
(e) 

3. Find the total length of each of the following simple closed curves by measuring each segment and adding all of the measures together. Use a ruler marked in 16ths of an inch.

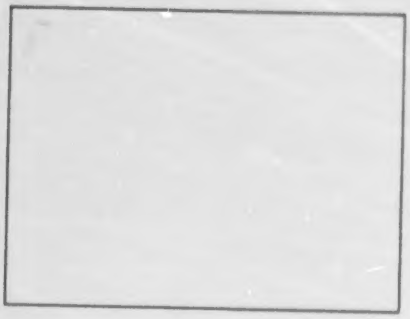
(a)



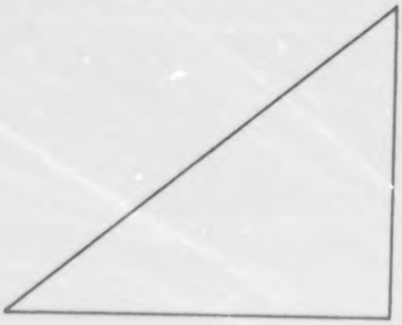
(b)



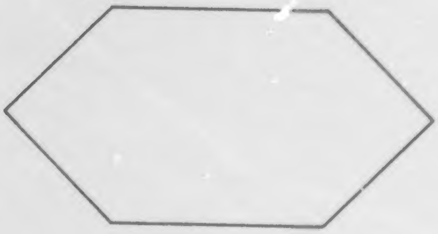
(c)



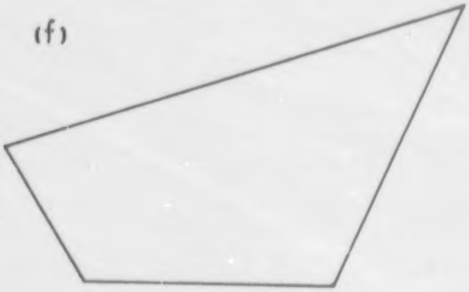
(d)



(e)

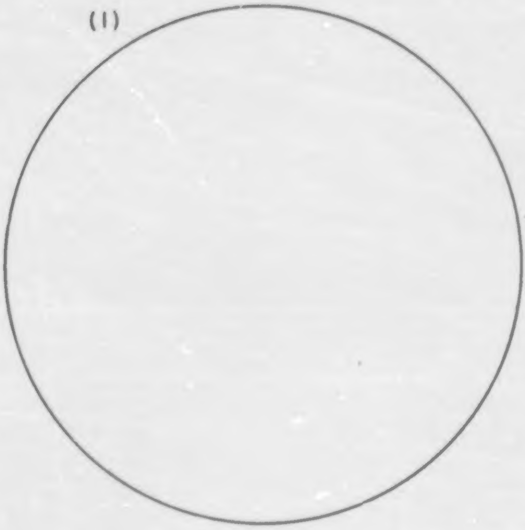


(f)

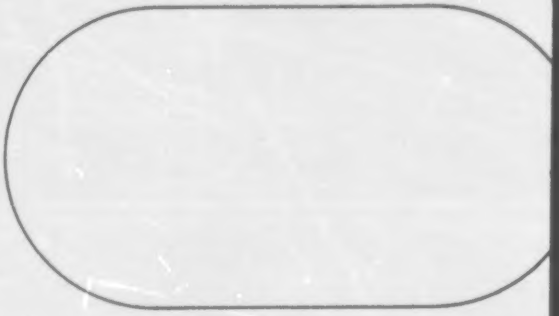


4. (a) In Problem 3 you used a ruler. Can you measure the length of each of the following closed curves in the same manner?
   (b) Can you think of a method for measuring the lengths by using string?
   (c) Measure the length of each closed curve below:

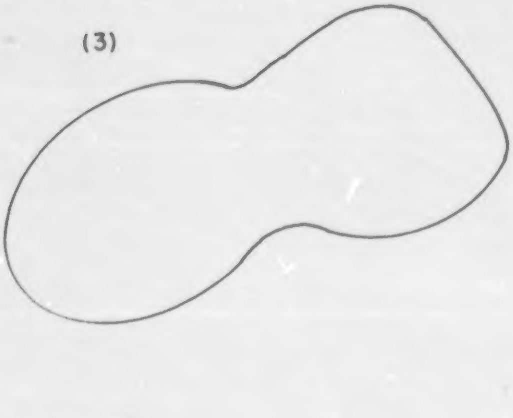
(1)



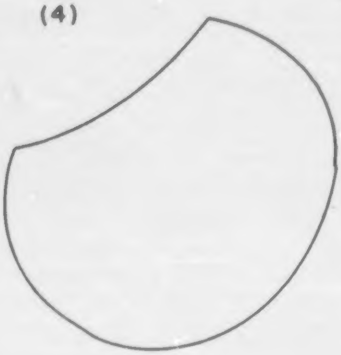
(2)



(3)



(4)

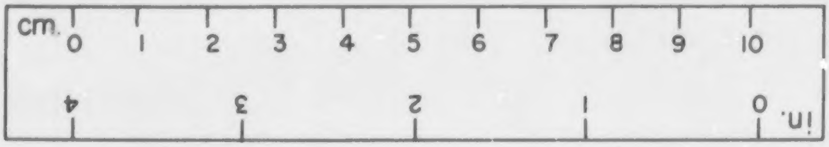


5. (a) Use 18 inches as the length of a cubit and find the dimensions of Noah's Ark. You can find the dimensions in cubits on page 261.
   (b) Give the dimensions of the Ark in feet and also in yards.

*6. Use an almanac to find the lengths of some modern ships. How do they compare with the reported length of the Ark?

Another Kind of Ruler

In the first part of this section you learned that the meter is the basic unit of length in the Metric System. However, it is frequently convenient to use another unit to measure lengths which are smaller than one meter. This unit is the centimeter. The centimeter is $\frac{1}{100}$ of a meter. Perhaps you own a commercially made ruler with one edge labeled in inches and the other edge labeled in centimeters. If not, the following picture will help you see what a unit of one centimeter looks like.

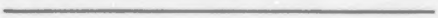


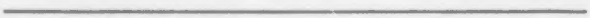
$$1 \text{ centimeter} = \frac{1}{100} \text{ of a meter}$$

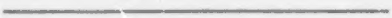
A length 100 times the size of the centimeter is the meter.

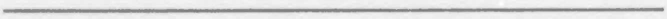
Class Exercises 7-4e

1. Use a piece of tagboard or cardboard, and a straightedge to make a 10 centimeter ruler.
2. Measure each of the line segments to the nearest centimeter.

(a)


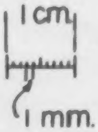
(b)


(c)


(d)


3. Find the total length in centimeters of the simple closed curves of Problem 4 of Exercises 7-4d.

Under many laboratory conditions it is frequently necessary to use a unit of measure smaller than the centimeter. For these purposes scientists use a unit of measure $\frac{1}{10}$ of a centimeter, called a millimeter. Notice that $\frac{1}{10} \times \frac{1}{100}$ of a meter is $\frac{1}{1000}$ of a meter. The prefix "milli" means one-thousandth. The following picture will help you see what a unit of 1 millimeter looks like. It is obtained by dividing the centimeter into 10 equal parts. Each of these parts is called a millimeter.



1 millimeter = $\frac{1}{10}$ of a centimeter = $\frac{1}{1000}$ of a meter.

A length 1000 times the size of the millimeter is the meter.

Class Exercises 7-4f

Use a commercial ruler marked in millimeters to perform the activities in this set of exercises.

1. Measure (in millimeters) each of the line segments in Problem 2 of Exercises 7-4e.
2. Find the total length in millimeters of the simple closed curves of Problem 3 of Exercises 7-4d.
3. (a) Draw a line segment 1 inch in length.
   (b) Measure this length with a centimeter ruler. About how many centimeters are there in one inch?
   (c) Measure the length of 1 inch in millimeters. About how many millimeters are there in one inch?

7-5. Precision of Measurement and the Greatest Possible Error

In working Problems 3-6 in Exercises 7-4b, you found different measures for the same segment when you used rulers with different size divisions. You have made rulers marked in inches, in $\frac{1}{2}$ in., in $\frac{1}{4}$ in., in $\frac{1}{8}$ in., and in centimeters. If you measure a segment with a ruler that is marked in 16ths of an inch, your measurement should be closer to the real length of the line segment than when you use any of the rulers you have made. You can buy rulers that are marked in 32nds or even 64ths of an inch. Since these have to be made very carefully, they are expensive and not in general use. If you used one of these rulers to measure the same segment, you might decide that the measurement you found previously was not quite the same as the new one. When you work with a ruler that has only inches marked on it, you should give your answer only to the nearest whole inch. If your ruler has $\frac{1}{2}$ inches marked on it, you can measure to the nearest $\frac{1}{2}$ inch. How precisely can you measure with a ruler marked in 8ths. A ruler marked in 16ths?

Class Discussion Exercises 7-5a

Turn to Exercises 7-4b, page 264.

1. Use your half-inch ruler to measure segment (b) in Problem 3.
2. Use your $\frac{1}{4}$" ruler to measure the same line segment.
3. Use your $\frac{1}{8}$" ruler to measure this line segment.
4. Measure segment (b) using 16th of an inch divisions.
5. Were the results of measuring with different size subdivisions the same?
6. Repeat Problems 1-5 for segment (c).

Your answer to Problem 1 should have been $3\frac{1}{2}$". This means that the right end of the line segment lies closer to the $3\frac{1}{2}$" mark than to either the 3" mark or the 4" mark. In Problem 2 you probably were not sure whether you should say $3\frac{1}{4}$" or $3\frac{1}{2}$" because the endpoint seemed to be halfway between the $3\frac{1}{4}$ mark and the $3\frac{1}{2}$" mark.

7. Fill in the blanks. In Problem 3 the right end of the segment is closer to ? than to either ? or ? . The measurement is stated to the nearest ? inch.

Even though you measured the same segment in Problems 1 to 4 above, your results were different. These differences occurred because you were using different subdivisions of an inch when you made the measurements. The measurement in Problem 1 is said to be precise to the nearest $\frac{1}{2}$ inch; in Problem 2 it is precise to the nearest $\frac{1}{4}$ inch. In Problem 3 the precision is to $\frac{1}{8}$ inch, while in Problem 4, the precision is to $\frac{1}{16}$ inch. Precision depends on the size of the smallest subdivision used on the measuring instrument.

We say that

measurement to the nearest $\frac{1}{2}$ inch } is more precise than { measurement to the nearest inch

measurement to the nearest $\frac{1}{4}$ inch } is more precise than { measurement to the nearest $\frac{1}{2}$ inch

and so forth.

Of course, anyone can make a mistake in reading the marks on a ruler or can be careless in placing it along the segment that is being measured; this is not what is meant by precision of measurement.

Let us write the subdivision sizes in the order in which we used them,

$$\frac{1}{1}\text{ in.},\quad \frac{1}{2}\text{ in.},\quad \frac{1}{4}\text{ in.},\quad \frac{1}{8}\text{ in.},\quad \frac{1}{16}\text{ in.},\ \ldots$$

and look at the denominators of the fractions,

1 2 4 8 16

Note that the

denominators increase in this direction ⟶
precision of measurement increases in this direction ⟶

The idea of precision of measurement is related to the idea of increasing denominators.

A $\frac{1}{4}$" ruler gives greater precision than a $\frac{1}{2}$" ruler.
A $\frac{1}{8}$" ruler gives greater precision than a $\frac{1}{4}$" ruler.

Suppose you wanted to measure the thickness of a hair. Would you use a ruler marked in inches? in $\frac{1}{8}$ inches? in $\frac{1}{16}$ inches? You probably said that these measuring tools were too "coarse," that you needed a "finer" instrument. Let us say that we need a more precise instrument. Do you know what a "precision instrument" is? The term applies to a wide variety of tools and instruments that are made with such great care that "fine" measurements can be made. This care is essential if the instrument is to be used for very precise measures. If very precise measures are needed to make a machine run properly, that machine is also called a precision instrument. A watch is one precision instrument. The family auto is another; it would not work very well if one of the cylinders were $\frac{1}{16}$ of an inch too large and, likewise, the piston would not even go into the cylinder if the cylinder were that much too small. Machinists often must make measurements correct to the nearest thousandth or ten-thousandth of an inch. This is such a small measurement that your eye cannot recognize it, but there are instruments with which such lengths can be recognized. One such precision instrument is the micrometer pictured in Figure 7-5a. Perhaps someone in the class can bring one and demonstrate how it works.

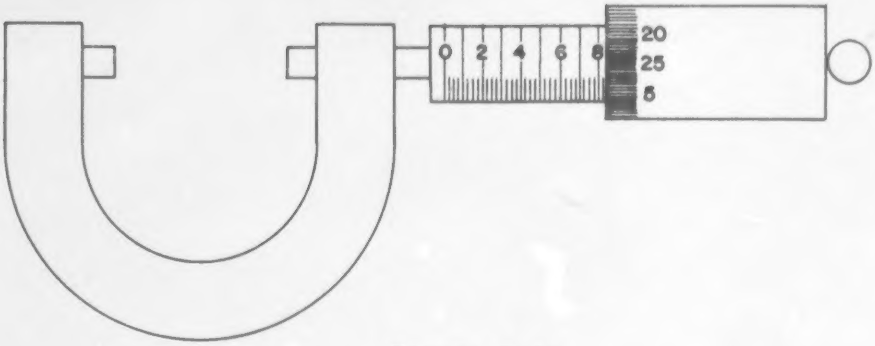


Figure 7-5a

Suppose we used a procedure for subdividing an inch unit which is in line with a scientist's thinking and directly related to our decimal system. The procedure would be something like the following:

A. Start with an inch unit. $\frac{1}{1}''$

B. Divide it into 10 equal parts. Thus, each part would be $\frac{1}{10} \times (1'') =$ $\frac{1}{10}''$

C. Divide the new $\frac{1}{10}''$ unit into 10 equal parts. Thus, each new part would be $\frac{1}{10} \times (\frac{1}{10}'') =$ $\frac{1}{100}''$

D. Divide the new $\frac{1}{100}''$ unit into 10 equal parts. Thus each new part would be $\frac{1}{10} \times (\frac{1}{100}'') =$ $\frac{1}{1{,}000}''$

E. Divide the new $\frac{1}{1{,}000}''$ unit into 10 equal parts. Thus each new part would be $\frac{1}{10} \times (\frac{1}{1{,}000}'') =$ $\frac{1}{10{,}000}''$.

Actually this is exactly the procedure that machinists follow in constructing the scale on a micrometer like the one pictured in Figure 7-5a. Suppose we have a ruler subdivided according to this plan. If we use only the $\frac{1}{10}$" marks we measure to the nearest $\frac{1}{10}$". If we use only the $\frac{1}{100}$" marks we measure to the nearest $\frac{1}{100}$". Suppose we arrange the denominators of our fractions in increasing order. We have

1   10   100   1,000   10,000

Note that

denominators increase in this direction →

precision of measurement increases in this direction →

Here too, you see, the idea of precision of measurement is related to the idea of increasing denominators.

When using fractional parts of a unit greater precision of measurement is obtained by using a subdivision whose fraction has a greater denominator.

A $\frac{1}{100}$" ruler gives greater precision than a $\frac{1}{10}$" ruler.

A $\frac{1}{1000}$" ruler gives greater precision than a $\frac{1}{100}$" ruler, etc.

From the preceding discussions it should be clear that as we progress to smaller units of measurement on the ruler we get better precision of measurement. Measurements made with a $\frac{1}{8}$" ruler are more precise than those made with a $\frac{1}{4}$" ruler. For the same reason measurements are said to be more precise if the ruler is graduated (marked) in:

inches	than if graduated in	feet
feet	than if graduated in	yards
yards	than if graduated in	rods
rods	than if graduated in	miles

For the same reason measurements will be more precise if the ruler is graduated in:

millimeters	than if graduated in	centimeters
centimeters	than if graduated in	meters

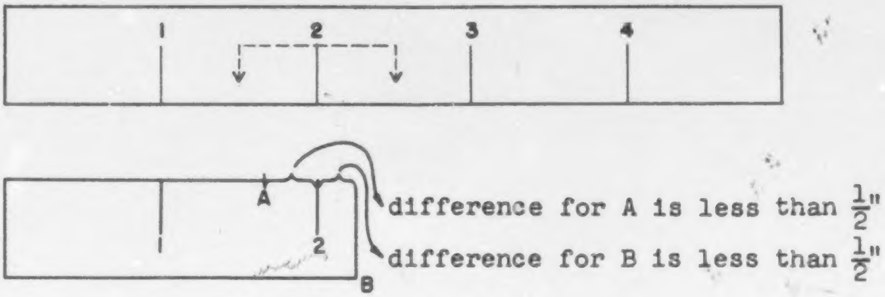
How can the notation of a measurement be written so that anyone looking at it can tell how precise the measurement is? One way is very simple--do NOT change the fractions to lowest terms. If you measure with a ruler that is marked in 16ths of an inch and you are using all of the divisions, your answer must be expressed in 16ths. For example, if a length is found to be $2\frac{1}{2}$ inches, you should write it as $2\frac{8}{16}$ in. Of course, if you measure only to the nearest 8th, the answer would be written $2\frac{4}{8}$ in. This method is used even when the measurement appears to be a whole number; 2 inches would be written as $2\frac{0}{16}$ in. or as $2\frac{0}{8}$ in., depending on the unit size being used. This method of reporting indicates the precision of measurement. Hence $2\frac{8}{16}$" indicates precision to the nearest $\frac{1}{16}$ in.

Class Discussion Exercises 7-5b

1. If you use a ruler marked in 8ths of an inch to measure a segment,
   (a) You are measuring to the nearest _____ inch.
   (b) Could the measurement be given as $3\frac{7}{8}$"?
   (c) Could the measurement be given as $3\frac{14}{16}$"?
   (d) How precise is a measurement obtained by using the 8th-inch ruler?
2. What would a measurement of $3\frac{14}{16}$" mean in terms of precision?
3. If you use a ruler marked in 16ths of an inch to measure a segment,
   (a) You are measuring to the nearest __?__ inch.
   (b) If your measurement is given as $2\frac{10}{16}$", should this number be written $2\frac{5}{8}$"? Why?
   (c) How precise is a measurement obtained?

4. Which would give the more precise measurement, a ruler marked in 8ths of an inch or a ruler marked in 16ths of an inch?

Now that we have some notion of precision of measurement, we should consider another idea which will help us to understand it more fully. Assume that you are using a ruler marked in inches and you give the measurement of a segment as 2 inches. This means that the right endpoint of the segment lies between $1\frac{1}{2}$" and $2\frac{1}{2}$". What is the greatest possible difference between the real length of the segment and your answer? Could this difference be larger than $\frac{1}{2}$"? The following diagram shows that whether the right endpoint lies at A or at B the difference between the endpoints and mark 2 is not larger than $\frac{1}{2}$".



difference for A is less than $\frac{1}{2}$"

difference for B is less than $\frac{1}{2}$"

Figure 7-5b

Class Discussion Exercises 7-5c

1. If your ruler is marked in $\frac{1}{2}$ in., the greatest possible difference between the true length and the measurement is __?__ in.

2. If your ruler is marked in $\frac{1}{8}$ in., the greatest possible difference is __?__ in.

The greatest possible difference between the real length of a segment and the measurement stated is called the greatest possible error. If you write a measurement as $2\frac{3}{8}$", the real length lies between $2\frac{5}{16}$" and $2\frac{7}{16}$". The greatest possible error is $(\frac{1}{2} \times \frac{1}{8})$ in. or $\frac{1}{16}$ in.

3. (a) Draw a diagram of a ruler similar to the drawing in Figure 7-5b, only marked in $\frac{1}{2}$ in.
   (b) Show on this diagram a segment whose measurement would be written $2\frac{1}{2}$".
   (c) Indicate the greatest possible error in making such a measurement.

4. Using a ruler marked in $\frac{1}{4}$ in., the greatest possible error is __?__ in.

5. When using a ruler marked in $\frac{1}{16}$ in., the greatest possible error is __?__ in.

6. Assume that the following measurements are written correctly (this means that even where possible, fractions are not simplified). Fill in the blanks as shown in the example.

	Measurement	Precision of Measurement	Greatest Possible Error
(a)	$3\frac{2}{4}$"	nearest $\frac{1}{4}$"	$\frac{1}{2} \times \frac{1}{4}" = \frac{1}{8}"$
(b)	$1\frac{15}{16}$"		
(c)	$4\frac{3}{8}$"		
(d)	$2\frac{6}{8}$"		
(e)	$3\frac{10}{16}$"		
(f)	$7\frac{4}{8}$"		
(g)	$2\frac{3}{32}$"	nearest $\frac{1}{32}$"	$\frac{1}{2} \times \frac{1}{32} = \frac{1}{64}"$

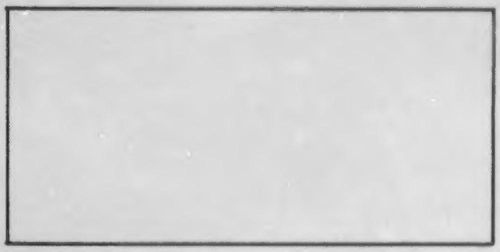
In Problem 6a above the greatest possible error is $\frac{1}{8}$". This does not mean that you made a mistake (or that you did not make a mistake) but that the real measurement lies between $(3\frac{2}{4} - \frac{1}{8})$ in. and $(3\frac{2}{4} + \frac{1}{8})$ in. The real measurement lies between $3\frac{3}{8}$" and $3\frac{5}{8}$".

7. For each measurement given in Problem 6, state between what two lengths the real measurement lies. The answer for Part (a) would be written: The real measurement lies between $(3\frac{2}{4} - \frac{1}{8})''$ or $3\frac{3}{8}''$ and $(3\frac{2}{4} + \frac{1}{8})''$ or $3\frac{5}{8}''$.
   The measurement for Part (g) would be written: The real length lies between $(2\frac{3}{32} - \frac{1}{64})''$ and $(2\frac{3}{32} + \frac{1}{64})''$.
   The real length lies between $2\frac{5}{64}''$ and $2\frac{7}{64}''$.

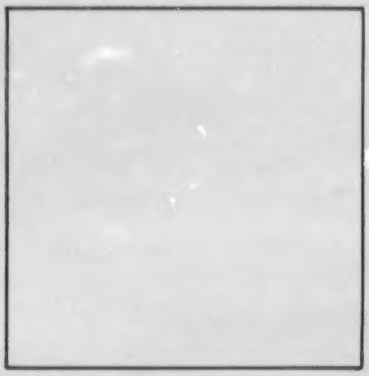
Exercises 7-5a

1. The greatest possible error is what fractional part of the smallest division used on the measuring instrument?
2. Draw a segment 2 inches long and divide it so that it can be used to show a precision of measurement of $\frac{1}{8}$ inch.
3. Draw a segment 2 inches long and divide it so that it can be used to show a greatest possible error of $\frac{1}{8}$ inch.
4. Measure the length and width of each of these figures to (1) the nearest $\frac{1}{2}$ inch; (2) the nearest $\frac{1}{8}$ inch; and (3) the nearest 16th inch. In each case give the greatest possible error.

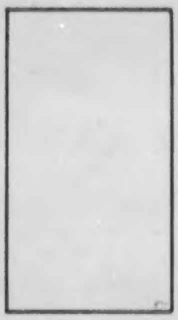
(a)



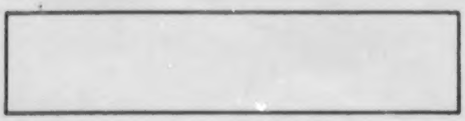
(b)



(c)



(d)



5. A rectangle has a length of 4 inches and a width of 3 inches. Each measurement is given with a precision of $\frac{1}{2}$".
   (a) Draw a rectangle using the longest possible segments that have these measurements.
   (b) Inside the rectangle in (a) draw a rectangle that has the shortest possible segments with these measurements.
6. A square has sides of $3\frac{1}{2}$ inches with a greatest possible error of $\frac{1}{8}$ inch.
   (a) Draw the largest square whose side has this measurement.
   (b) Inside the first square, draw the smallest square with this measurement.

Another way to indicate the greatest possible error is often used by machine shops and by engineers. For this we need a new symbol, a plus sign over a minus sign which looks like this, " ±," and is read "plus or minus." For example, if the ruler is marked in 16ths of a inch and you use all of the divisions, a line segment 2 inches long would be written as "$2 \pm \frac{1}{32}$ in." This means that the segment might be as long as $2\frac{1}{32}$ in. or as short as $1\frac{31}{32}$ in. and still have the name "2 inches." Look back at Figure 7-5b to make the meaning of this notation clear. Notice that both the measurement, 2", and the greatest possible error, $\frac{1}{32}$", appear in this notation. Since the greatest possible error is one-half the subdivision size on the ruler used, then the subdivision size is twice the greatest possible error. Hence, for our example the size of the measuring unit on the ruler is $2(\frac{1}{32}") = \frac{1}{16}"$. The measurement is precise to the nearest $\frac{1}{16}$th inch.

Exercises 7-5b

1. Draw a line and mark on it a scale with divisions of $\frac{1}{4}$ inch. Mark the zero point C, place a point between $1\frac{2}{4}$ inch and $1\frac{3}{4}$ inch but closer to $1\frac{3}{4}$ inch, and call the point D. How

long is $\overline{CD}$ to the nearest $\frac{1}{4}$ inch?

2. Express the length of $\overline{CD}$ using the greatest possible error notation.

3. Write the length of $\overline{CD}$ using the notation of precision to show the size of the divisions on the line you drew.

4. Between what two points on the scale must D lie if the measurement, to the nearest $\frac{1}{4}$ inch, is to be $1\frac{3}{4}$ inches? How far from $1\frac{3}{4}$" is each of these points?

5. (a) The measurement of a line segment was stated to be $1\frac{2}{8}$".
   This segment must have been measured to the nearest __?__ of an inch.

   (b) The endpoint of the segment must have fallen between __?__ and __?__ .

   (c) The greatest possible error in the measurement of this segment is __?__ .

   (d) Express this same measurement using another type of notation.

6. (a) The measurement of a line segment was stated to be $(2\frac{1}{4} \pm \frac{1}{16})$ in. This line must have been measured to the nearest __?__ of an inch.

   (b) The endpoint of the segment must have fallen between __?__ and __?__ .

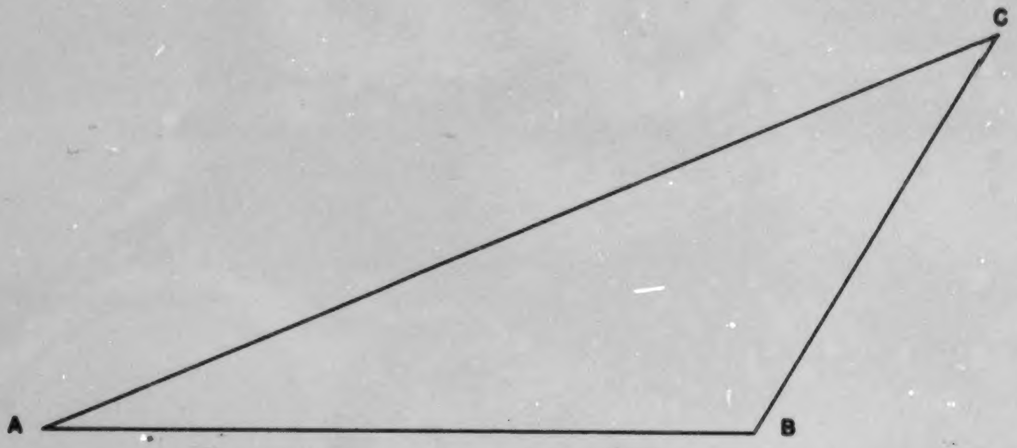
   (c) The greatest possible error in the measurement of this segment is __?__ .

   (d) Express the measurement in (a) in another way.

7. Measure the length of your notebook and express the measure using the idea of the greatest possible error to indicate the size of the divisions on your ruler.

8. Measure the length of your notebook and express the measure using the idea of precision to indicate the size of the divisions on your ruler.

9. Measure the lengths of each side of the triangle and express your answer in two ways.



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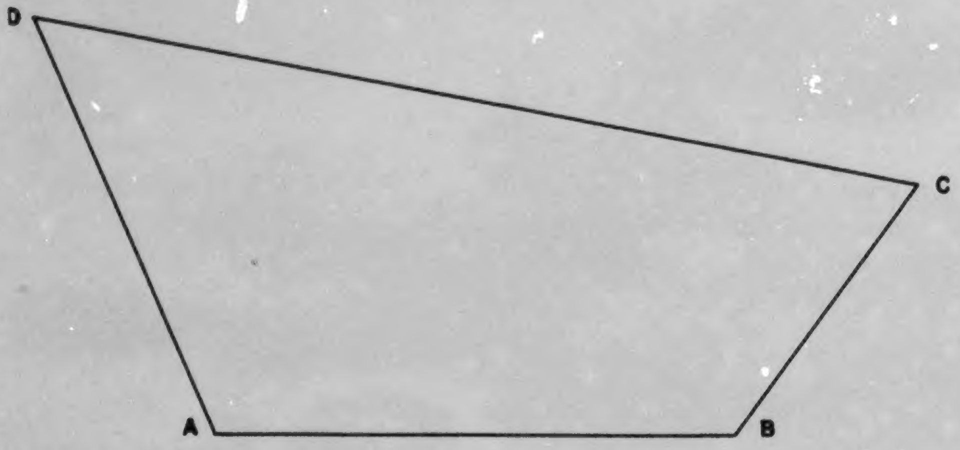
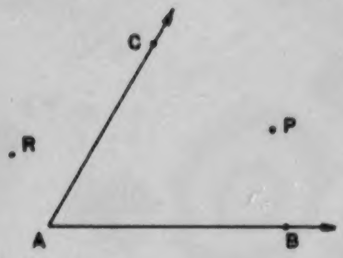


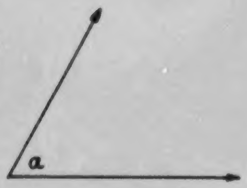
Measure the length of each side of the quadrilateral and express your answer in two ways.

7-6. Measurement of Angles

You have been studying ways of measuring line segments, plane closed regions, and solids. Now let us see how angles are measured.

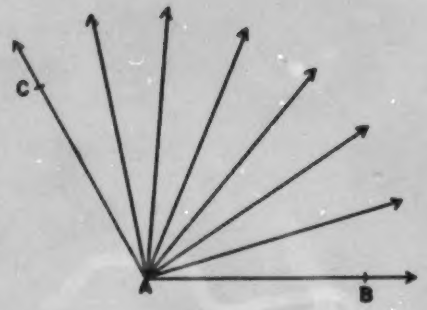
Recall that an angle is the set of points on two rays with the same endpoint. In the drawing of the angle, the rays are $\overrightarrow{AB}$ and $\overrightarrow{AC}$. These rays are the sides of angle BAC and point A is its vertex. Notice that the angle is named "angle BAC" or "angle CAB," with the vertex named second. Why should A be the middle letter? We can call it "angle A" if this name applies to only one angle. We also name angles by writing a small letter or numeral in the interior of the angle, near the vertex.

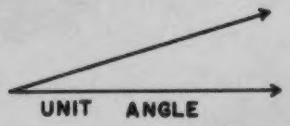




An angle determines three sets of points in the plane, the set of points in the interior of the angle, the set of points in the exterior of the angle, and the set of points on the angle itself. A point P is in the interior of angle BAC if it is on the same side of line $\overleftrightarrow{AB}$ as point C, and on the same side of line $\overleftrightarrow{AC}$ as point B. (See angle BAC above.) Any point in the plane which is not a point on the angle and not a point in the interior is a point in the exterior of the angle. Is point R a point in the interior or the exterior of angle BAC? Point P?

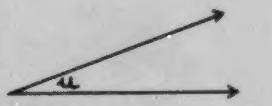
As you know, to measure anything you must use a unit of the same nature as the thing measured. To measure an angle, an angle is chosen as the unit of measure. Then an angle can be measured by drawing rays which subdivide its interior so that angles are formed which are exactly like the unit angle. In the sketch, the interior of angle BAC is subdivided so that angles are formed which are exactly like the unit angle shown. So the size of $\angle$BAC is 7 times the size of the unit angle.

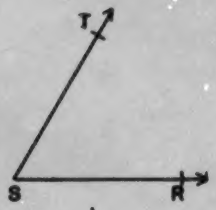




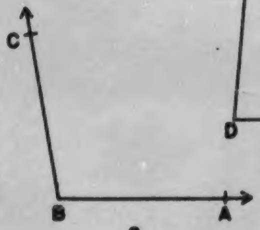
Exercises 7-6a

Copy the angles below on thin paper, and also the unit angle u. Cut out the angular region determined by the unit angle and use it to subdivide the interior of each of the angles.





1.



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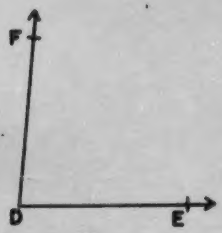
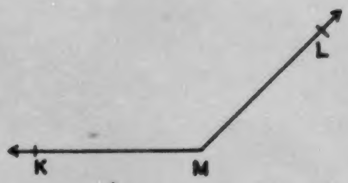


3.



4.

1 to 4. Compare each angle with the unit angle, stating your comparisons like this: The size of angle RST ≈ _____ u.

Standard Unit for Angles

Just as there are standard units for measuring a line segment (inch, foot, yard, millimeter, centimeter, meter) so are there standard units for measuring an angle. The one we shall use is determined by a set of one hundred eighty-one rays drawn from the same point. These rays determine 180 congruent angles which, together with their interiors, make a half-plane and the line which determines the half-plane. The rays are numbered in order from 0 to 180, forming a scale. To each ray corresponds a number; that is, there is a number for each ray, and a ray for each number from 0 to 180. Not all 181 rays are shown in the sketch below, but the ray corresponding to 0 and every tenth ray thereafter is drawn. One of these 180 congruent angles is selected as the standard unit. The measurement of this angle is called a degree. The measure of this unit angle, in degrees, is 1.

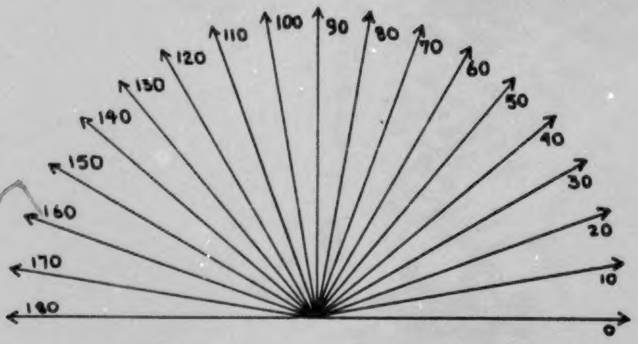


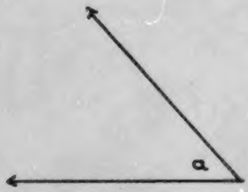
Figure 7-6a

You can use a scale like this to measure an angle. Place the angle on the scale with one side of the angle on the ray that corresponds to zero and the other side on a ray that corresponds to a number less than 180. The vertex of the angle is placed at the intersection of the rays. Then the number which corresponds to that ray is the measure of the angle, in degrees. The size or measurement of the angle is that number of degrees.

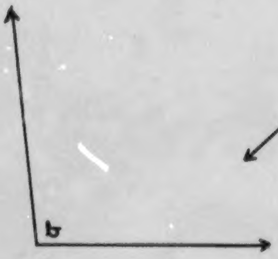
The symbol for "degree" is "$^{\circ}$". Thirty-five degrees may be written "35°".

Exercises 7-6b

Copy the angles below by tracing them. Then measure each angle by placing it on the scale in Figure 7-6a.



1.



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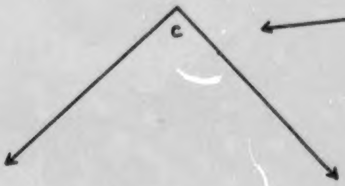

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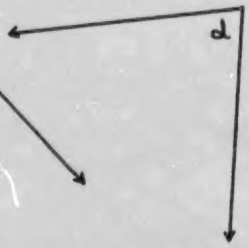
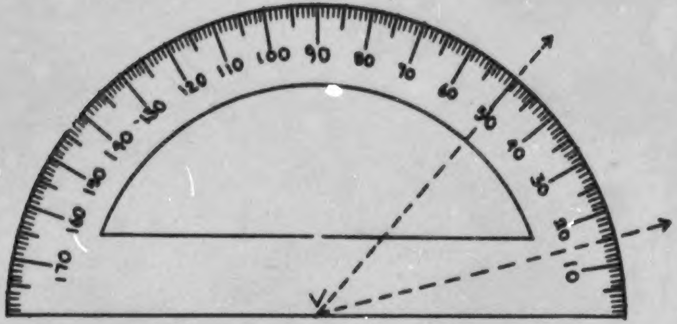


4.

The Protractor. The method you used for the exercises above is inconvenient, so usually an instrument called a protractor, is used. Then the scale can be placed on the angle, rather than the angle on the scale.

Look at this drawing of a protractor. Think of the rays from point V. Segments of these rays are shown on the curved part of the protractor. In the drawing, two of the rays are shown in dotted lines, to show how to think about them. These



rays correspond to the numbers from 0 to 180, and every ray named by a multiple of 10 is labeled. To measure an angle with the protractor, place the protractor on the angle so that point V is on the vertex of the angle and the ray which corresponds to zero on the protractor lies on one side of the angle. Then observe the protractor ray which is on the other side of the angle. The number that corresponds to this ray is the measure, in degrees, of the angle.

You will find that your protractor has two scales (only one of these is shown on the diagram). One scale starts with zero at the right and runs to 180 at the left. The other scale starts with zero at the left and runs to 180 at the right. When you read the measure of an angle, you must be sure to read the same scale that shows zero for one side of the angle.

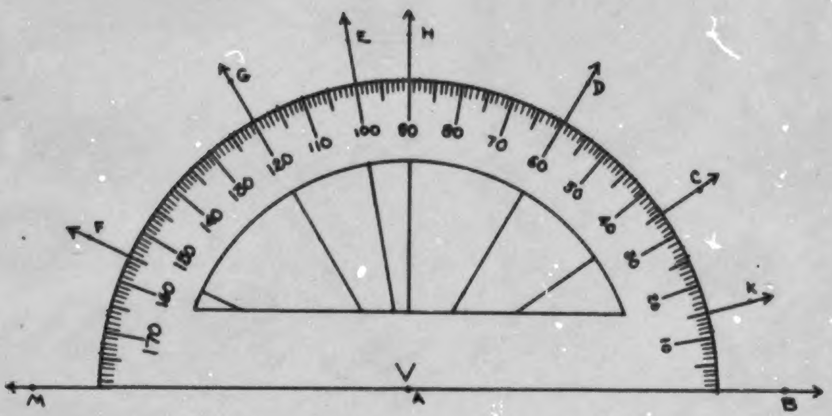
Exercises 7-6c

1. [illegible] the drawing on page 292 is shown a protractor placed on a [illegible]gure with several rays drawn from point A. Find the measure, in degrees, of each of the angles.

   (a) $\angle BAK$ (b) $\angle BAC$ (c) $\angle BAD$ (d) $\angle BAH$ (e) $\angle BAE$

   (f) $\angle MAF$ (g) $\angle GAM$ (h) $\angle MAC$ (i) $\angle DAE$ (j) $\angle CAG$

   (k) $\angle KAF$ (l) $\angle HAF$


E
H
G
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C
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K
10
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30
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50
60
70
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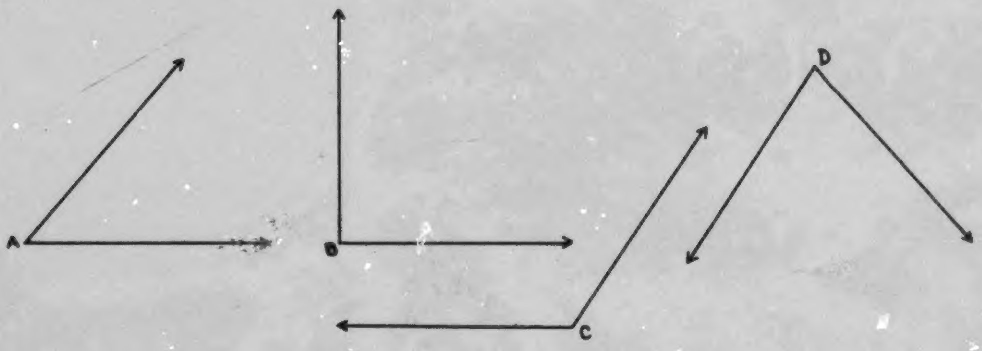
90
100
110
120
130
140
150
160
170
V
M
A
B

2. Use a protractor to measure the angles below. If the parts of the rays shown are not long enough to show the intersection of the side of the angle with the rim of the protractor, lay the edge of a piece of paper along the side of the angle.

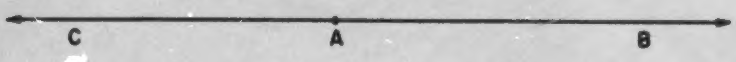


3. Draw a ray $\overrightarrow{AB}$ with endpoint A. Place your protractor with point V on A and the protractor ray which corresponds to 0 on $\overrightarrow{AB}$. Then mark the point at 35 on the protractor scale, and name the point C. Remove your protractor and draw ray $\overrightarrow{AC}$. You should now have an angle BAC. What is its measure?

4. Use the method described in Problem 3 to draw angles of these sizes:

   (a) 20° (b) 45° (c) 61° (d) 90° (e) 130°
   (f) 179°

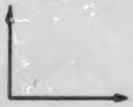
5. In the drawing below, ray $\overrightarrow{AB}$ and ray $\overrightarrow{AC}$ are opposite rays; that is, they are on the same line, they have the same endpoint, and their intersection set is the endpoint A.



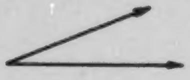
   (a) If the protractor is placed so that V is on A and the zero ray of one scale is on ray $\overrightarrow{AB}$, what number corresponds to the protractor ray on ray $\overrightarrow{AC}$?
   (b) Is CAB an angle? Why?

Sets of Angles. Angles may be separated into sets according to their measures.

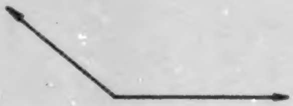
An angle of 90 degrees is called a *right angle*.



An angle whose size is less than 90 degrees is called an *acute angle*.



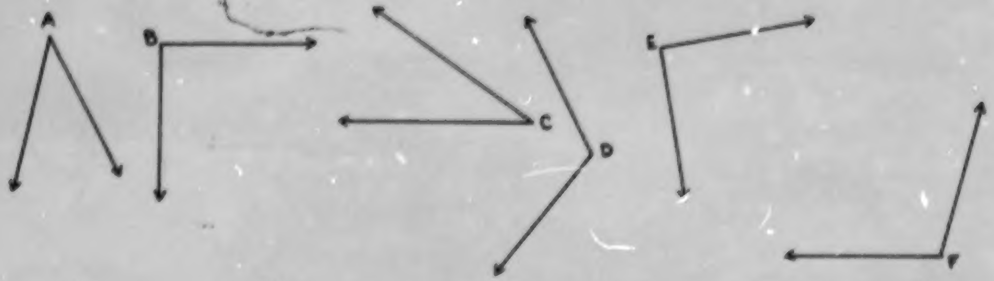
An angle whose measure is more than 90 and less than 180 is called an *obtuse angle*.



Exercises 7-6d

1. Without measuring, tell which of the angles on page 294 appear to be:

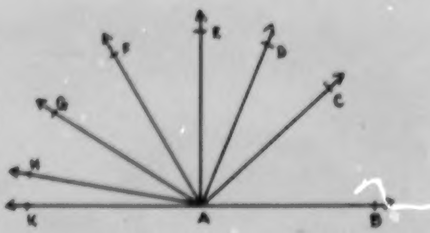
   (a) right angles; (b) acute angles; (c) obtuse angles.



2. Were you uncertain about any of the angles in Problem 1? If so, check your answer by using a protractor.

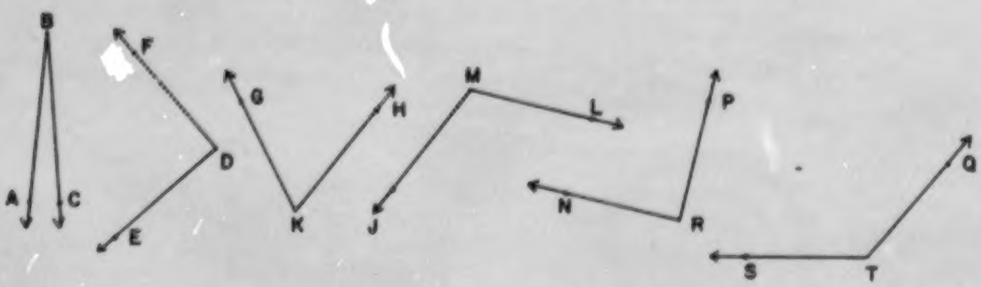
3. (a) The measure, in degrees, of an acute angle is greater than __?__ and less than __?__ .
   (b) The measure, in degrees, of an obtuse angle is greater than __?__ and less than __?__.

4. (a) In the figure, name all the obtuse angles which have ray $\overrightarrow{AB}$ as one side; all such acute angles; all such right angles.
   (b) Name all the acute angles that have ray $\overrightarrow{AE}$ as one side; all such obtuse angles; all such right angles.
   (c) Name all the right angles that have ray $\overrightarrow{AK}$ as one side; all such obtuse angles; all such acute angles.



5. (a) Without measuring, tell whether each angle below appears to be an acute, right, or obtuse angle.

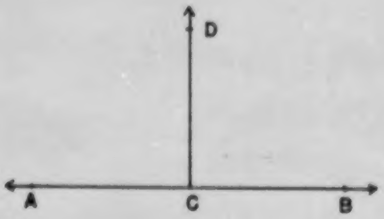
(b) Without measuring, estimate the number of degrees in the size of each angle. (A good way to do this is to think how it compares with a right angle.)



(c) Measure each angle in Problem 5b. How well did you estimate their sizes?

6. Find six physical representations of our idea of an angle--two acute, two right, and two obtuse angles.

Perpendicular Lines



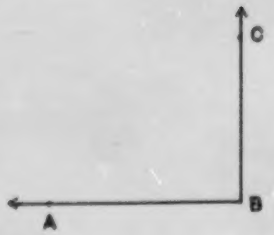
In the figure, ray $\overrightarrow{CD}$ is drawn from point C on line $\overline{AB}$, so that angle BCD and angle DCA have the same measure. We say that ray $\overrightarrow{CD}$ is perpendicular to line AB.

If a protractor is placed with point V on point C and the zero ray of the protractor on $\overrightarrow{CB}$, the protractor ray which falls on $\overrightarrow{CA}$ corresponds to __?__ . Therefore the size of angle BCD, and also of angle DCA is __?__ degrees, and they are both __?__ angles. So we may also say that a ray is perpendicular to a line if the ray and the line intersect so that at least one angle formed is a right angle.

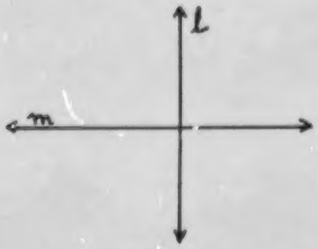
When two rays form a right angle we say the rays are perpendicular. Which rays in the figures for Problem 5 of Exercises 7-6d seem to be perpendicular rays?

When two lines intersect, they are perpendicular if one of the angles determined by the lines is a right angle. Line segments are perpendicular if the lines on them are perpendicular.

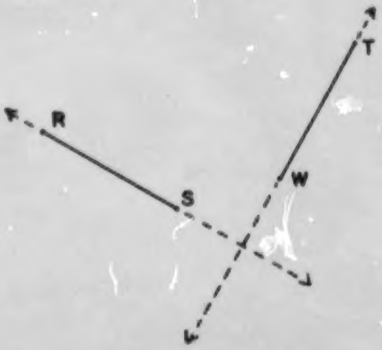
The symbol for "perpendicular" is " $\perp$ ".



$\overrightarrow{BA} \perp \overrightarrow{BC}$.

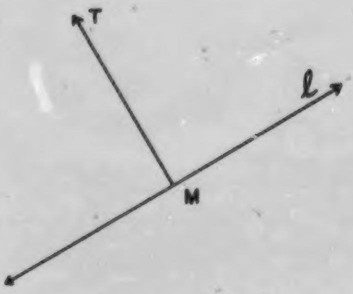


Line $\ell \perp$ line m.

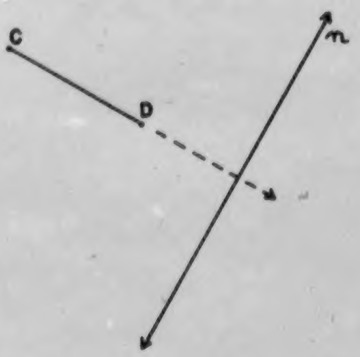


Segment $\overline{RS} \perp$ segment $\overline{TW}$.

We may also say a line is perpendicular to a ray, or a line segment is perpendicular to a ray or a line.



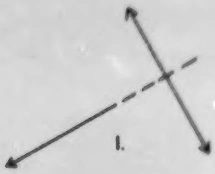
Line $\ell \perp$ ray $\overrightarrow{MT}$.

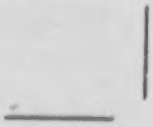


Segment $\overline{CD} \perp$ line n.

Exercises 7-6e

1. Which of the drawings below represent
   (a) Perpendicular lines?
   (b) Perpendicular rays?
   (c) Perpendicular line segments?
   (d) A line perpendicular to a ray?
   (e) A ray perpendicular to a line segment?
   (f) A line segment perpendicular to a line?


1.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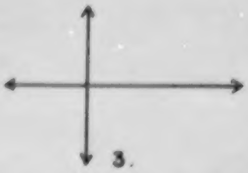
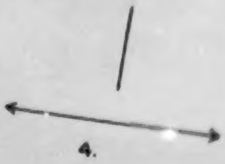
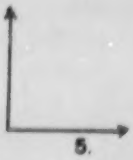
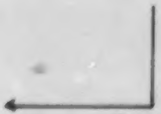

3.


4.


5.


6.

2. Find five physical representations of pairs of perpendicular rays, lines, or line segments.

3. Find five physical representations of pairs of rays, lines, or line segments which are not perpendicular. If they intersect, tell whether the rays on them form an acute angle or an obtuse angle.

7-7. Summary

1. The sizes of collections of separate objects may be found by counting, but the sizes of continuous quantities are found by measuring.
2. Measurment is approximate, not exact; and when possible, the precision or greatest possible error in a measurement should be shown.

3. The symbol ≈ means "is approximately equal to."

4. Measurement of the continuous geometric quantities of length, angle, area, and volume may be thought of as a process of "covering" with units of a given size.

5. When geometric continuous quantities are measured, the unit used must be of the same kind as the quantity measured, i.e., a unit segment to measure segments, a unit angle to measure angles.

6. The size of units of measurement is entirely arbitrary, but in practice it is essential to have standard units which are agreed upon by large groups of people.

7. A ruler may be used as a number scale to measure segments, and a protractor may be used as a number scale to measure angles.

Chapter 8

AREA, VOLUME, WEIGHT, AND TIME

8-1. Rectangle

We will consider here the most familiar of the simple closed curves, the rectangle. Let us agree that a rectangle is a four-sided figure (in a plane) which has a right angle at each of its four corners. Is the cover of your book in the shape of a rectangle? Find five examples of rectangles in your classroom. Under this definition is a square a rectangle?

In Sections 3 and 4 of Chapter 7 you found the lengths of closed curves by measuring the segments forming the curves. You have also measured closed regions determined by the curves. What instrument did you use to measure the lengths of the sides? Why did you need a new kind of unit to measure a closed region?

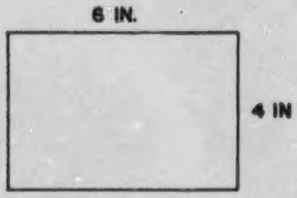
The total length of a closed curve is called its perimeter. For a rectangle, this is the distance an ant would walk if it started at one corner of the rectangle and kept walking along the sides till it returned to its starting point.

Exercises 8-1a

1. Measure the four edges of the front cover of this book. Pick out two additional rectangles and measure the four sides of each.
2. Find the perimeters of each of the three rectangles in Problem 1.
3. In any rectangle there are two pairs of opposite sides, that is, sides that do not meet at a vertex. For each rectangle in Problem 1 write down the lengths of the pairs of opposite sides. From looking at these lengths complete the following statement:

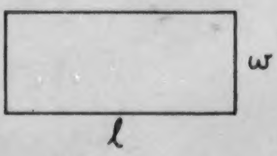
   The lengths of two opposite sides of a rectangle are ____________.

4. Suppose two sides of a rectangle are 6 inches and 4 inches as shown:



Use the result of Problem 3 to tell the lengths of the remaining sides. Find the perimeter of this rectangle.

5. If l and w stand for the number of units in the lengths of two sides of a rectangle, what are the numbers of units in



the lengths of the other sides? Write a number sentence which tells how to find the number of units in the perimeter if you know l and w.

The lengths of two intersecting sides of a rectangle are often called the length and width of the rectangle. Quite frequently in discussing other simple closed curves the total distance around it is called the length of the curve instead of the perimeter. However, as just noticed, for a rectangle the word "length" indicates the length of the longer side. We will use it here in this sense.

6. A school playground is rectangular, 400 ft. long and 200 ft. wide. What is the total length of the fence around the playground? Express the answer in feet and in yards.

7. If fencing costs $5 a yard, how much did the fence of Problem 6 cost?

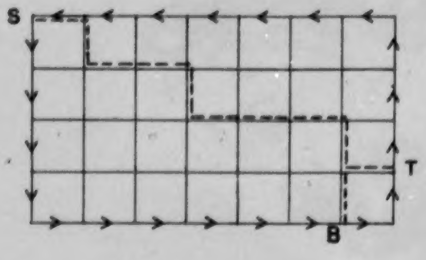
8. A carpenter is putting picture molding around a room which has a length of 15 ft. and a width of 10 ft. How much molding will he need? Express the answer in inches, in feet, and in yards.

9. The carpenter of Problem 8 is also putting a baseboard around the room. He observes that the room has four doorways, and that each doorway is 3 ft. wide. Since he does not put baseboard across the doorways, how many feet of baseboard will he need? (For your calculation does it matter where the doorways are located?)

10. A boy has 24 ft. of wire fence to make a rectangular pen for his pet rabbit. He plans to use all the fence in making the pen. Could he make a pen 12 ft. long and 12 ft. wide? Why or why not? Could he make a pen 8 ft. long and 3 ft. wide? How about 8 ft. long and 4 ft. wide? Give five examples of lengths and widths he could use for his pen. (Use only whole numbers for lengths and widths.)

11. A girl is decorating for a party. She has 5 tables, each 28 inches wide and 42 inches long, and wants to put a strip of crepe paper around the edge of each table. How many yards of crepe paper are necessary?

12. A Fourth of July parade is to follow a rectangular route as shown by the arrows, starting and ending at S, where the squares represent city blocks.



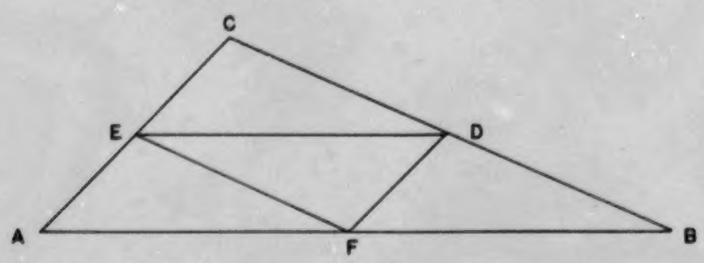
In this city each block is $\frac{1}{8}$ mile on a side. What is the total length of the parade route? Express the answer in at least two ways.

13. If the decorations along the parade route in the parade of Problem 12 cost about $250 a mile, what was the approximate total cost of the decorations?

14. In the parade of Problem 12 two men got tired and sneaked back to the starting point along the dotted lines. One man left the parade at point T and the other at B. How much distance did each man save?

*15. A farmer found that it took 240 feet of fence to go around his rectangular farmyard. He noticed that one of the sides is 40 feet long. How long is each of the other sides? Let x stand for the number of feet in the width and write a number sentence describing this problem.

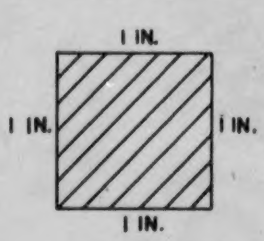
*16. You have worked with a figure like the one below. Here D, E, F are the midpoints of the sides. Let a stand for the number of units in the length of segment $\overline{AF}$. Make a copy of the figure and label with an a all the segments which have the same length as $\overline{AF}$. Similarly, let b be the number of units in the length of $\overline{AE}$ and c the number of units in the length of $\overline{BD}$. Label the other segments with measures of b or c. When you worked with this figure before, you found the 11 simple closed curves which are contained in the figure. Name each of the eleven curves and for each of the curves write a number sentence for the number of units in the perimeter (the total length).



Areas of Rectangles

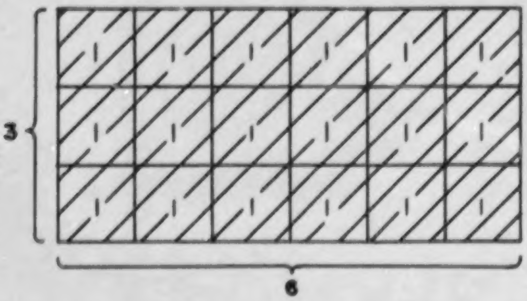
Let us now turn our attention to the closed region determined by the rectangle. As you learned earlier, to measure this you choose the closed region determined by some simple closed curve for a unit of area and then see how many of these units it takes to cover the closed region to be measured. This is often an awkward and laborious process. Why can't we have some simple instrument like a ruler that we can lay on the area and read off the answer at once?

In your earlier work with area you tried out the closed regions determined by several different simple closed curves as possible units of area. We would like now to choose a definite unit for area. The usual choice is a square closed region. On the basis of your experience in Problem 1 of Exercises 7-3 does this seem like a good choice? Why or why not? In addition we have a choice as to how large a square to use. To be definite let us take for the unit of area the closed region determined by a square whose side is one unit of length. Since we have used many units of length, we get correspondingly many units of area. If length is measured in inches, we have as corresponding unit of area, the closed region determined by the square shown below.



The area of this closed region is called a square inch. Describe and name three other units of area related to different units of length.

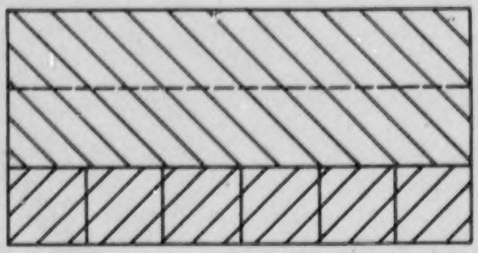
If a rectangle is 6 units long and 3 units wide, then it is clear at once that the rectangular closed region can just be covered by square units of area as shown.



The measure of the area is, then, by definition, the number of square closed regions. How many of them are there? Is there an easier way to get the number than by counting them? If so, how?

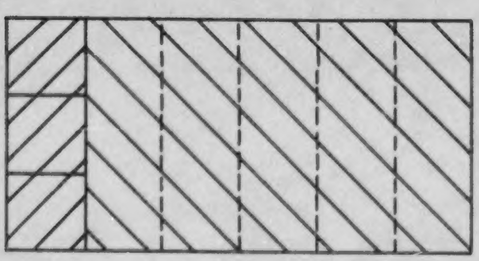
If you counted by observing that each row has six squares and that there are three rows, you obtained the number of square units A in the area by writing:

$$A = 3 \times 6$$



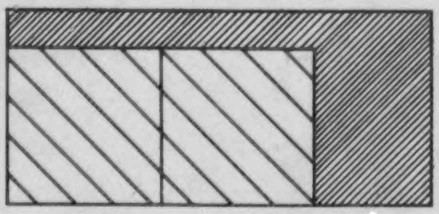
If you counted by saying that there are three squares in each column and six columns you obtained:

$$A = 6 \times 3$$

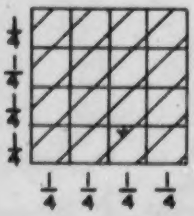


Are these answers the same? If so, what property of rational numbers does this illustrate? What advantage do you see in this discussion in having the unit of area chosen so that the sides of the square are each one unit of length? What if the unit of area were determined by a square whose side was $1\frac{1}{2}$ units?

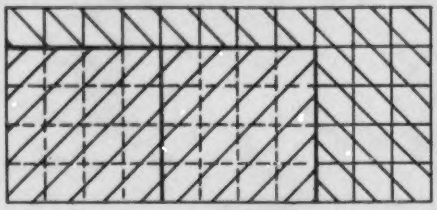
In practice it is much more likely that the measures of the sides will be rational numbers which are not whole numbers. Suppose for instance that the length is $2\frac{3}{4}$ inches and the width $1\frac{1}{4}$ inches. We can easily put in two square inch units, but are still left with a border as shown by the shaded area.



In order to fill this border conveniently let us take a unit square inch (or several if necessary) and cut it up into smaller square closed regions as shown.



This particular division is chosen because the measurements given for the rectangle are in quarter inches. How many of the small square closed regions are there in the square inch? What part of a square inch is each small square closed region? These small square units can be used conveniently to fill out the uncovered border above as shown.



As a matter of fact, if we cut up the original two square-inch units also, as shown by the dotted lines, the entire rectangular closed region to be measured is covered by these small square units.

We have considered the area of a closed region. Recall that a closed region consists of a simple closed curve and its interior. It would be just as possible to talk about the area of the interior of the simple closed curve. If such a discussion were carried through carefully (this may be done later in your mathematical experience), we would find that

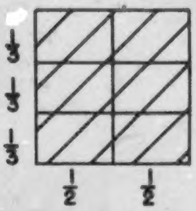
the area of the interior of a simple closed curve
= the area of the corresponding closed region.

Class Exercises 8-1a

1. In the above figure showing the rectangle $2\frac{3}{4}$ inches by $1\frac{1}{4}$ inches:
   The number of small squares in each row is ___?___; the number of rows is ___?___; the total number of small squares is ___?___; the area of each small square unit is ___?___ square inches. The area of the rectangle is therefore ___?___ square inches.

In the problem was it necessary to draw the figure to find how many squares were in each row? To find the number of rows? If not, how do you determine these numbers?

2. The length and width of a rectangle are given $5\frac{1}{2}$ inches and $4\frac{1}{2}$ inches. A small square unit of what size would be convenient in covering this rectangle? How many squares are in each row? How many rows are there? What is the number of square inches in the area? Draw a figure showing the small squares. Was it necessary to draw the figure to find the area?
3. Using the same method as above, find the area of a rectangle whose length and width are $5\frac{1}{2}$ inches and $4\frac{2}{3}$ inches. (In this case you may find it convenient to cut the unit square inch into closed regions rectangular in shape rather than square by making the division shown below.



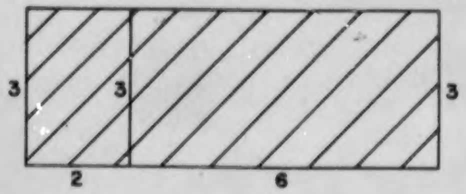
Why might this division occur to you? What part of the square inch is each small rectangular unit?

Exercises 8-1b

1. From your experience with the preceding problems, state in words a method by which you can find the number of square units in the area of a rectangle if you know the number of units in the length and width.
2. If l and w stand for the number of linear units in the length and width of a rectangle, and A is the number of square units in its area, write a number sentence telling how to find A if you know l and w. Notice that this is just the translation

into mathematical language of the statement in Problem 1 above.

3. How many square inches are in a square foot? How many square feet in a square yard? Draw figures to illustrate your answer. Make these figures actual size, either on paper or on a surface such as the blackboard. (Newspapers can be pasted together.)
4. Draw a 3 inch square, i.e., a square 3 inches on a side. Draw also a rectangle whose area is 3 square inches. Which is larger? What is the area of the square?
5. Draw two different rectangles, each of which has an area of 1 square inch. Make one rectangle 2 inches long and the other 4 inches long. (This illustrates how an area of 1 square inch may occur in many different shapes.)
6. A living room rug is 9 ft. by 12 ft. (a) Find its area. (b) Express the result in square yards.
7. A baseball diamond is a square, 90 ft. on a side. Find the area of the interior of this diamond both in square feet and in square yards.
8. The diamond used in softball is a square 60 ft. on a side. Is this area more or less than half that of a baseball diamond? (See Problem 7.)
9. Find the number of square yards in a square mile.
10. Two rectangles are placed together as shown so that a larger rectangle is formed, with the number of linear units in the sides as indicated. Find the areas of both smaller rectangles and of the larger one in square units. Is the larger area the sum of the other two? Write a number sentence determined by this relationship among the three areas. What property of rational numbers does this number sentence illustrate?



[sec. 8-1]

11. A rectangle is 3 units long and 2 units wide. If another rectangle is twice as long but has the same width, how do the areas of the two rectangles compare? Draw a figure illustrating your answer. Do the same if the new rectangle has the same length as the original one but twice the width.

12. Does the reasoning of the last problem depend on the particular measures, 3 and 2? If not, write a statement telling the effect on the area of doubling the length of any rectangle. On doubling the width.

13. If a rectangle has length and width of 3 units and 2 units, what is the effect on the area of doubling both length and width? Draw a figure to illustrate your conclusion. If the reasoning does not depend on the particular rectangle, write a statement telling the effect on the area of doubling both length and width of any rectangle.

14. In the rectangles of Problem 13 compare the two perimeters. Write a statement telling the effect on the perimeter of a rectangle of doubling both length and width.

15. A rectangle has length 313 inches and width 211 inches. Its area, by Problem 2, is 313 × 211 square inches. (Do not do the multiplication.) If a new rectangle is twice as long and twice as wide, its area is 626 × 422 square inches. Without multiplying, show that

$$626 \cdot 422 = 2^2 \cdot 313 \cdot 211.$$

What property or properties of rational numbers did you use? What does the statement say about the areas of the two rectangles? Does this agree with your conclusion in Problem 13?

*16. By reasoning similar to that of Problem 14, find the effect on area if the length and width of a rectangle are tripled. Draw a figure illustrating the conclusion. What is the effect on the area if the length is doubled and the width tripled?

*Precision and Error

In the discussion so far it has been assumed that the exact lengths and widths of the rectangles are known. Actually of course, we have seen this is never the case since no measurement can be made exactly. Thus if we have measured a rectangle and found measurements of $2\frac{1}{2}$ inches and $3\frac{1}{4}$ inches, we must use the "approximately equal" symbol and write $\ell \approx 3\frac{1}{4}$, $w \approx 2\frac{2}{4}$ and therefore:

$$
\begin{aligned}
A &= \ell w \\
A &\approx (3\tfrac{1}{4})\ (2\tfrac{1}{2}) \\
A &\approx (\tfrac{13}{4})\ (\tfrac{5}{2}) \\
A &\approx \frac{65}{8} \\
A &\approx 8\tfrac{1}{8}
\end{aligned}
$$

Since A is the number of square inches, we find therefore that the area is approximately $8\frac{1}{8}$ square inches.

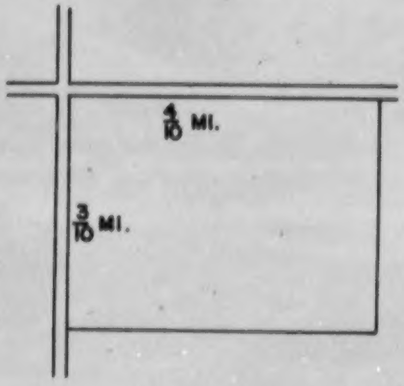
A statement concerning a measured quantity should indicate that it is only approximate.

Exercises 8-1c

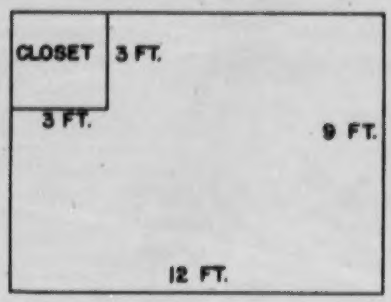
Use the $\approx$ sign in connection with numbers representing measured quantities.

1. Measure the length and width of the top of your desk to the nearest half inch.
   (a) Find the number of square inches in the area.
   (b) What is the perimeter?
2. A section of chalkboard is about 5 feet long and $3\frac{1}{2}$ feet wide.
   (a) Find the area. Express the answer in three ways.
   (b) Find the perimeter and express it in three ways.
3. A rectangular field is located at the intersection of two perpendicular roads. Using the mileage indicator of a car, the length and width are measured as approximately $\frac{3}{10}$ mile and $\frac{4}{10}$ mile. Find the area of the field, and express the

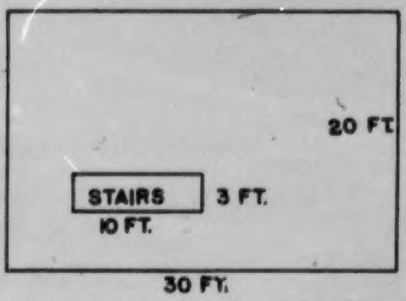
result in at least two different units.



4. A rectangular lawn is found to be 84 feet by 50 feet. The lawn is being seeded with grass, and the directions on the box of grass seed say that one pound of seed is enough for 300 square feet. How many pounds of grass seed are necessary?
5. A bathroom floor is tiled with small tiles which are closed regions one inch square. The floor contains 3240 of these tiles. What is the area of the floor in square yards?
6. The floor of a boy's room is in the shape of a rectangle. The length and width are 12 feet and 9 feet. There is a closet 3 feet long and 3 feet wide built in one corner, as shown in the floor plan. What is the floor area of the room (outside the closet)?



7. The floor of an attic is in the shape of a rectangle with measurements of 30 feet by 20 feet. There is an opening in the floor as shown where the stairway comes up. Find the actual floor area of the attic. Express the result in two forms. In finding the floor area does it matter where the opening for the stairs is placed?



*8. A farmer has had a rectangular garden for a number of years. He knows that the length of wire fence around it is 500 feet and has found by experience that he uses a 100-pound bag of fertilizer on it each year. One spring he decides to enlarge his garden so it will be twice as long and twice as wide. Since the old fence is worn out anyway he throws it away. He then goes to the hardware store and orders 1000 feet of fencing and 2 hundred-pound bags of fertilizer. Is this order reasonable? Why or why not?

Class Exercises 8-1b

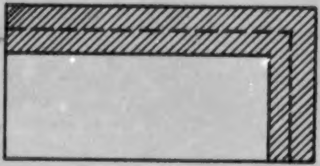
1. The length and width of a rectangle, R, have been measured as $3\frac{1}{4}$ inches and $2\frac{3}{4}$ inches so that $\ell \approx 3\frac{1}{4}$ and $w \approx 2\frac{3}{4}$. Draw a rectangle with these dimensions and find the area of its interior by using the result of Problem 2 of Exercises 8-1b. This area will be called the calculated area.

2. (a) If it is understood in Problem 1 that $\ell \approx 3\frac{1}{4}$ means that the length was measured to the nearest quarter inch, then all we are sure of is that the number of inches in the true length is somewhere between __?__ and __?__.

   (b) Similarly, the number of inches in the true width is between __?__ and __?__.

   (c) On the figure you made for the last problem show the largest and smallest lengths which are described by $\ell \approx 3\frac{1}{4}$.

(d) Do the same for the width. Then draw in the largest and smallest rectangles which could be correctly described by $\ell \approx 3\frac{1}{4}$, $w \approx 2\frac{3}{4}$, where ℓ and w stand for number of inches. The area between these two rectangles represents the uncertainty in the correct area and is shaded in the diagram. The true area lies between that of the smallest rectangle and that of the largest rectangle.



3. (a) The area of the smallest rectangle in Problem 2 is __?__ sq. in.

   (b) The area of the largest rectangle in Problem 2 is __?__ sq. in.

   (c) The difference between the calculated area of R and the answer to (a) above is __?__ sq. in.

   (d) The difference between the calculated area of R and the answer to (b) above is __?__ sq. in.

The answers to the above questions can be assembled in a table.

Minimum Rectangle	Measured Rectangle	Maximum Rectangle	
$3\frac{1}{8}$ in.	$3\frac{1}{4}$ in.	$3\frac{3}{8}$ in.	Length
$2\frac{5}{8}$ in.	$2\frac{3}{4}$ in.	$2\frac{7}{8}$ in.	Width
$\frac{525}{64}$ sq. in.	$\frac{572}{64}$ sq. in.	$\frac{621}{64}$ sq. in.	Area

Also,

difference in part (c) = calculated area - minimum possible area,

$$\frac{47}{64} \text{ sq. in.} = \left(\frac{572}{64} - \frac{525}{64}\right) \text{ sq. in.}$$

and

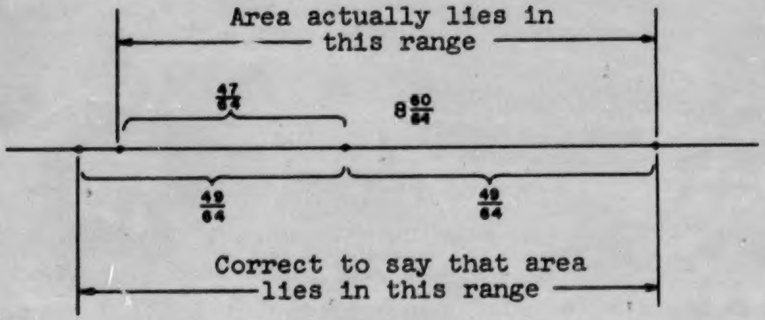
difference in part (d) = maximum possible area - calculated area,

$$\frac{49}{64} \text{ sq. in.} = \left(\frac{621}{64} - \frac{572}{64}\right) \text{ sq. in.}$$

Thus, the true area of our rectangle R lies between

$$\left(8\frac{60}{64} - \frac{47}{64}\right) \text{ sq. in.} \quad \text{and} \quad \left(8\frac{60}{64} + \frac{49}{64}\right) \text{ sq. in.}$$

The computed area of rectangle R, as found in Problem 1 is $8\frac{60}{64}$ or $2\frac{3}{4} \times 3\frac{1}{4}$. The calculated area of R can be too large by as much as $\frac{47}{64}$ sq. in. or too small by as much as $\frac{49}{64}$ sq. in. on the other side. The greatest possible error for the computed area of R is $\frac{49}{64}$ sq. in. Hence, it would be correct to say that the true area lies between $(8\frac{60}{64} - \frac{49}{64})$ sq. in. and $(8\frac{60}{64} + \frac{49}{64})$ sq. in. Notice that the numbers in parentheses lie an equal distance on each side of $8\frac{60}{64}$. These ideas are illustrated on a number line.


Area actually lies in this range
47/64
8 60/64
49/64
49/64
Correct to say that area lies in this range

Now we can indicate the precision of the calculated area $(3\frac{1}{4} \times 2\frac{3}{4})$ sq. in. by writing

$$\text{True area} = \left(8\frac{60}{64} \pm \frac{49}{64}\right) \text{ sq. in.}$$

This means that the true area will not vary from $8\frac{60}{64}$ sq. in. by more than $\frac{49}{64}$ sq. in.

In Problem 1 the answer obtained by multiplication gave an area expressed in sixteenths of a square inch. This would suggest that the answer is correct to the nearest sixteenth of a square inch. Is this true according to the results obtained in this problem?

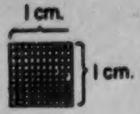
The last three problems illustrated an important fact. When an area is obtained after multiplying the measures of an approximate length and width, the possible error in the area is much greater than the form of the answer would suggest. Thus when in our work we write area $\approx 8\frac{15}{16}$ sq. in, we will not mean that this has a precision of $\frac{1}{16}$ square inch.

Exercises 8-1d

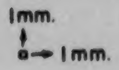
Find the calculated area for each rectangle whose dimensions are given below; then determine the precision of your answer and indicate it by using the greatest possible error notation.

1. $3\frac{1}{2}$ in. by $4\frac{1}{2}$ in.
2. $1\frac{3}{4}$ in. by $2\frac{1}{4}$ in.
3. $2\frac{3}{8}$ in. by $3\frac{4}{8}$ in.

In the previous sets of exercises you used the foot, the inch, etc. in finding perimeters and areas of closed regions. Now let us make measurements using some of the metric units. You recall the metric units for linear measurements were the meter, centimeter $= \frac{1}{100}$ (meter) and the millimeter $= \frac{1}{1000}$ (meter). The corresponding metric units for area measurements are square closed regions having edges which measure 1 meter, 1 centimeter, and 1 millimeter respectively. The diagram on the left on the following page is a picture of a one centimeter square closed region; the one on the right is a one millimeter square closed region.



Area = 1 sq. cm.



Area = 1 sq. mm.

Exercises 8-1e

1. How many square millimeters are there in a square centimeter?
2. How many square centimeters are there in a square meter?
3. How many square millimeters are there in a square meter?
4. Draw a 3 cm. square. Draw also a rectangle whose area is 3 square centimeters. Which is larger?
5. A rug is 2 meters by 3 meters. Find its perimeter and area.
6. The floor of a boy's room is in the shape of a rectangle. The length and width are measured as 4 meters and 3 meters. There is a closet 1 meter long and 1 meter wide built in one corner. What is the floor area of the room (outside the closet)?

8-2. Rectangular Prism

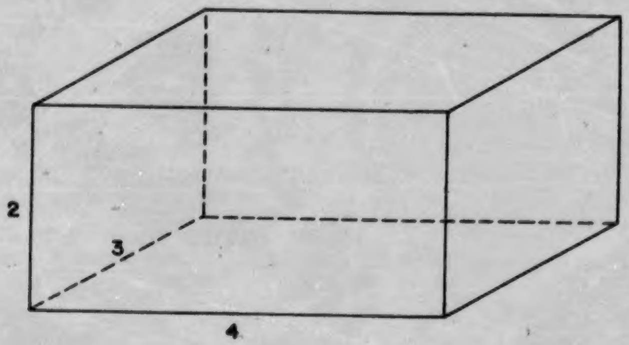
A figure shaped like a chalk box will be referred to as a rectangular prism. It is one of the most familiar figures and you will find many examples. Your classroom is probably one example. Name as many examples as you can. There will be further discussion of prisms and other figures later in your course. When you walk across the classroom floor you are moving in the interior of the rectangular prism of your classroom, if your room has this shape. Let us examine such a prism. Note that the prism has a certain number of flat sides. These are called its faces. How many faces does a rectangular prism have? What kind of figure is each of the faces? Notice that each of the faces lies in a plane. For each face of the prism notice that there is just one other face that does not meet it. Such a pair of faces is called opposite faces.

Opposite faces actually lie in parallel planes. How many pairs of opposite faces are there? Identify the pairs of opposite faces in your classroom. What can you say about the shape of two opposite faces? How do you know?

You learned in Chapter 4 that two planes which intersect must intersect in a line, so two faces which are not opposite must intersect in points that lie on a line. Actually they meet in the points of a line segment. You recall that these segments are called the edges of the prism. How many edges are there on a rectangular prism? Identify them in your classroom. Some of these edges have the same length. (What sets of edges are equal? Why?) How many different lengths could there be among the edges? There are on the prism certain points where three faces intersect, or what amounts to the same thing, where three edges intersect. These points are called vertices of the prism. One such point is called a vertex. How many vertices are there? Point out the vertices of your classroom.

As you have probably noticed, the parallel edges have the same length, for example, the edges from the bottom to the top, or the edges from one end to the opposite end. There can then be at most three different lengths.

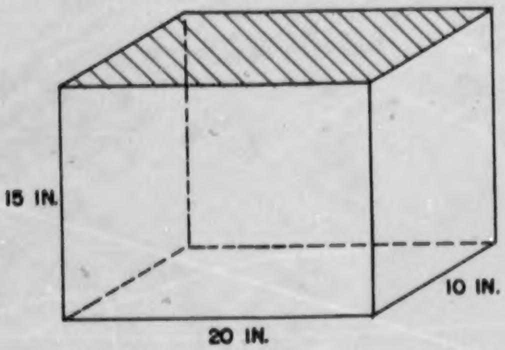
In the figure below, the number of units in the lengths of three edges have been marked. How many units are there in the length of each of the other nine edges.



The lengths of the edges in the three possible directions are often called the length, width, and height of the prism. What do you notice about the pairs of opposite faces? Since all the faces are rectangular closed regions, it is easy to find all their areas. The sum of the areas of all the faces is called the surface area of the rectangular prism.

Exercises 8-2a

1. Find the surface area of the rectangular prism just discussed.
2. A housewife has a cake tin in the shape of a rectangular prism (without a top). The tin is 10 inches long, 8 inches wide, and 2 inches deep. In baking a cake she lines the pan with wax paper. How many square inches of wax paper are necessary to line the tin exactly?
3. A classroom wall is 30 feet by 10 feet. On this wall is a strip of chalkboard 20 feet long and $3\frac{1}{2}$ feet wide. The wall is to be painted except for the chalkboard. Compute the area to be painted. Express the answer in square feet, then in square yards.
4. An incubator for hatching eggs is in the shape of a rectangular prism 20 inches long, 10 inches wide, and 15 inches high. The top is made of glass (indicated by the shading) while the



sides and bottom are made of wood. What is the area of the top? What is the area of the outside surface of the wooden part of the box? Express your answers both in square inches and square feet.

5. A room is 15 feet long, 12 feet wide, and 9 feet high.
   (a) How many asphalt tiles, each 12 inches by 12 inches, will be necessary to tile the floor?
   (b) How many tiles will be necessary if each is 6 inches by 6 inches?
6. In the room of Problem 5 there are five windows in the walls, each 3 feet wide and 6 feet high. How much wall surface is there, not counting the windows? In finding your answer does it matter where the windows are placed?)
7. In Problem 6 how many quarts of paint are necessary to paint these walls if a pint of paint will cover 66 square feet?
8. A trunk is 3 feet long, 18 inches wide, and 2 feet high. The edges are all reinforced with strips of brass. How much brass stripping is necessary? Express the answer in inches, in feet, and in yards.
9. A cube is a rectangular prism for which all the edges are congruent so all the faces are square closed regions. How many square inches of wood are needed to make a covered cubical box with edges of 18 inches each? How many square feet?

*10. Let ℓ, w, h stand for the numbers of units in the length, width, and height of a rectangular prism. Write a number sentence telling how to find the number, S, of square units in the surface area.

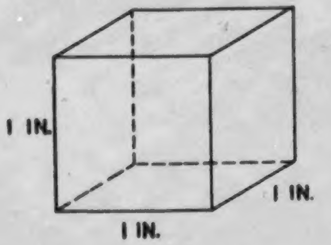
*11. If ℓ, w, h have the same meanings as in Problem 10, write a number sentence telling how to get the total number, E, of units of length in all the edges.

*12. A manufacturer makes tool boxes (with covers) which are $2\frac{1}{2}$ feet long, 1 foot wide, and 6 inches high. He has 50 such boxes and wants to varnish the outsides except for the bottoms. If a pint of varnish will cover the closed rectangular region 9 feet by 8 feet, will two quarts of varnish be enough to varnish the tool boxes?

Volume

The term rectangular solid will refer to the set of points consisting of a rectangular prism and its interior. We wish to find the volume of this rectangular solid. We will refer to this volume as the volume of the prism. In Section 7-3 you discussed measuring volume by picking out some convenient unit of volume and seeing how many such units were necessary to make up the solid. Measuring volumes by this process, while showing what we mean by measuring, has serious drawbacks in practice. Imagine trying it on such a problem as finding the volume of your classroom. It would certainly be useful if we had a way of finding the volume of a rectangular prism by working with the length, width, and height just as we learned to find the area of a rectangle by working with its length and width. First, however, it is necessary to agree on a unit of volume. The usual choice for a unit of volume is a cubical solid. A cube is a rectangular prism for which all the edges are congruent. Would this have been your choice, or would you have chosen something else? What size cube would you recommend? Why?

The usual choice is a cubical solid, each edge of which is a unit of length. In this case what can be said about the size of the faces? It is the relationship among the units of volume, area, and length which makes it easy for us to figure out the volumes. If we choose to measure lengths in inches then the unit of volume would be a cubical solid, each edge of which is 1 inch. The volume of this solid is called a cubic inch.

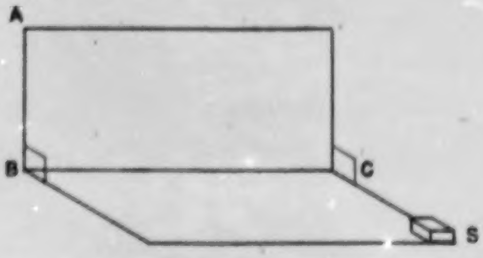


As in the case of the rectangle, where we talked about the area of its interior, we can talk about the volume of the interior of a rectangular prism. It is the same as the volume of the corresponding rectangular solid.

Class Exercises 8-2

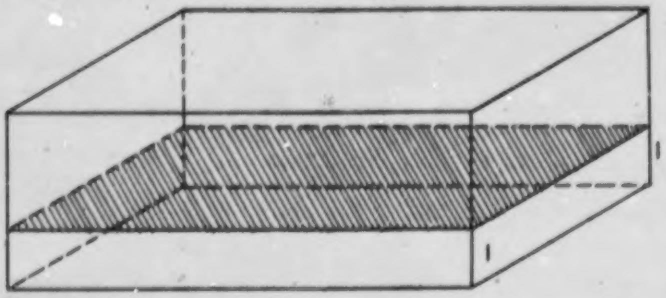
1. Describe and name at least three units of volume.
2. Obtain some small cubical blocks of the same size and think of the edge as a unit of length. Put these together to form a rectangular solid 4 units by 3 units by 2 units. How many blocks did it take?
3. (a) Use small cubical blocks to make a solid 3 units by 2 units by 2 units.
   (b) Make a rectangular solid with the length twice as long but the height and width the same. (6 by 2 by 2.)
   (c) Make a rectangular solid with the width twice as long as the original but the other measures the same. (3 by 4 by 2.)
   (d) Make a rectangular solid with the height twice as long as the original but the other measures the same. (3 by 2 by 4.)
   (e) Compare the number of blocks in (b), (c) and (d) with the number in (a).
4. Write a statement telling the effect on the volume of a rectangular prism if one of the measurements is doubled.
5. (a) Build a rectangular solid 6 units by 4 units by 2 units. Notice that here both the length and width of the prism are the doubles of the length and width in Problem 3 (a).
   (b) What is the ratio of the number of blocks in the new solid to the number in the original (Problem 3-a)?
   (c) Would you get the same ratio if you doubled a different pair of measurements?
6. Write a statement telling the effect on the volume of a rectangular prism if two of the measurements are doubled.

7. (a) Double each edge of the solid in Problem 3 (a) and construct a rectangular solid. (6 units by 4 units by 4 units.)
   (b) What is the ratio of the number of blocks in this solid to the number in the original solid?
8. Write a statement telling the effect on the volume of a rectangular prism if each edge is doubled.
9. BRAINBUSTER. Two rectangular metal plates are welded together at right angles as shown:



An ant at A wishes to crawl along the plates to get the lump of sugar at S. How should he crawl to get to S in the shortest distance?

Volume of a Rectangular Prism



Since we know how to find the area of any face, let us imagine we have already found the area of the face on which the solid is shown to be resting. Suppose the area of this bottom face (often called the base) is 12 square units. If the base consists of 12 unit square closed regions, let us place a unit cubical solid on each of these regions. These unit solids fill up a layer one unit thick across the bottom of the solid. Since there are 12 such cubical solids, the volume of the layer is 12 cubic units. You saw examples of such layers in the problems in the last section. Since the top of the layer is just like the bottom, a second layer can be laid on the first and so on. If the height of the prism is 3 units, then it will be exactly filled with 3 layers, and the number, V, of units of volume will then be given by

$$V = 3 \times 12 = 36$$

Thus the volume is 36 cubic units.

If the measure of height is a rational number, which is not a whole number, such as, $2\frac{1}{3}$ units, then two layers will not fill it completely, but three layers will be too much. In fact, if we slice the third layer horizontally, we need to use only the bottom third of the layer. The volume of this is $(\frac{1}{3})(12)$ cubic units, and the total volume of the prism is $2(12) + (\frac{1}{3})(12)$ cubic units, or $(2\frac{1}{3})(12)$ cubic units, or 28 cubic units.

We have assumed that the base was exactly made up of whole units. What would you do if the base were broken up into parts of units, as would happen if the base were 8 units by $1\frac{1}{2}$ units? Do you come to the same conclusion about the volume? Why?

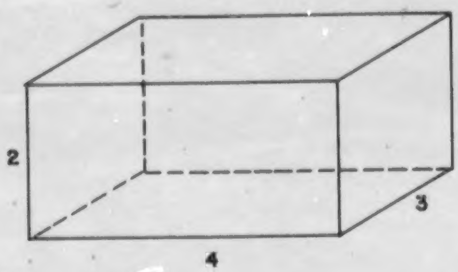
Exercises 8-2b

1. Find the volume of the interior of a closet $8\frac{1}{2}$ feet high if the floor is 10 square feet.
2. The base of a child's sandbox is a rectangular closed region, and its area is 24 square feet. If the box is 10 inches deep, find its volume.
3. Write a statement telling how to find the number of cubic units in a rectangular prism if the number of square units in the base and the number of units in the height are known.
4. How deep should the sandbox of Problem 2 be made if the box is to hold 48 cubic feet?

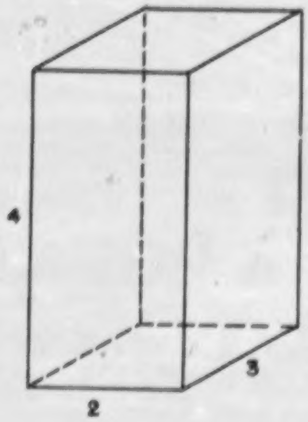
5. A man is making a wooden box to hold 260 cubic feet of sand. In order to fit into a certain space the box must be 13 feet long. How many square feet of area must be in the end of the box? Write a number sentence that describes this problem. Also express the answer to the problem in square inches and in square yards.
6. Health regulations in a certain school district say that the school rooms must contain 50 cubic feet of air for each child. If a room has a floor area of 160 square feet and is 10 feet high, can the principal legally put 30 children in it? What is the greatest number of children who may legally be assigned to it?
7. A rectangular box is h units high and has a base whose area is B square units. Write a number sentence showing how to get the number, V, of cubic units of volume in its interior if the numbers h and B are known. Notice that this is just a mathematical sentence for the statement you made in Problem 3.

Look at the work of the last problem. Do you need to know the exact shape of the base? What must you know in order to find the volume of a rectangular prism. This procedure of finding the volume of a prism from the area of the base and height without needing to know the exact shape of the base will be used again when you consider other prisms and cylinders.

If we know all the edges of a rectangular prism, we know how to find the area of the base, so we can easily get the volume.



In a rectangular prism 4 units by 3 units by 2 units, we probably would think of the base as the largest face. The area of this face is 4 × 3 square units so the volume, by Problem 4 above, would be 2 × (4 × 3) cubic units. Notice that the number 4 × 3 which is enclosed in parentheses is the number of square units of area in the base. If we stand the prism on end (or turn our necks through a right angle) we think of another face as being the base.



We find that the area of the base is 2 × 3 square units and the volume of the solid is 4 × (2 × 3) cubic units. Resting the prism on its third face, we find that the volume is 3 × (2 × 4) cubic units. Since it is the same solid in different positions we seem to be claiming that

$$2 \times (4 \times 3) = 4 \times (2 \times 3) = 3 \times (2 \times 4)$$

Is this true? If not, where did our discussion miss the boat? If true, what property or properties of rational numbers does it illustrate?

Exercises 8-2c

1. The bottom of a desk drawer will just take two pieces of ordinary typewriter paper laid end to end. (Typewriter paper is $8\frac{1}{2}$ inches by 11 inches.) If the drawer is 4 inches deep find its volume. If the bottom and sides of this drawer are to be lined with wax paper so as to fit exactly, how many square inches of wax paper will be needed?
2. A woman has some blankets to store. She estimates that when folded they would require 10 cubic feet of space. Could she store them in a trunk 3 feet long, 18 inches wide and 2 feet high? If the trunk had the same length and height but was 20 inches wide, could she store the blankets?
3. Write a statement in words telling how to obtain the number of cubic units in the volume of the interior of a rectangular solid if the number of units in the length, width, and height are known.
4. If ℓ, w, h, stand for the number of units in the length, width, and height of a rectangular prism, write a number sentence telling how to find the number of cubic units, V, of volume in the interior of the solid.
5. How many cubic inches are there in a cubic foot? How many cubic feet are in a cubic yard? Show how you obtain these conclusions.
6. (a) In a 3-inch cube (a cube each edge of which is 3 inches), what is the volume of the cube?
   (b) Is it larger or smaller than 3 cubic inches? Be very careful not to confuse the volume of a 3-inch cube with a measured volume of 3 cubic inches.
   (c) If ℓ is the number of inches in the edge of a cube, is it always true that the volume of an ℓ inch cube is greater than ℓ cubic inches?
7. If a rectangular prism is 2 inches long and 1 inch wide, what is its height if the rectangular solid is to have a volume of 1 cubic inch? Make a paper model of this prism to illustrate a different possible shape for a volume of 1 cubic inch.

8. Is the volume of the interior of a rectangular prism 59 inches long, 37 inches wide, and 23 inches high given by this: $(23 \cdot 37 \cdot 59)$ cubic inches? (Do not multiply.) Write an expression like that above for a rectangular prism with measurements just twice those in the first prism. Factor this expression; if any factor occurs more than once, write it with an exponent. The second volume is how many times the first volume?

*9. Using reasoning like that of Problem 8, tell what will be the effect on the volume of a rectangular prism if all its measurements are tripled? What will be the effect if two are doubled and one tripled?

The volumes obtained here by multiplication would be exactly correct if the length, width, and height were exactly correct. Since these quantities are always approximate, we should actually use the "approximately equal" sign $\approx$ in writing statements about measurement. In the remaining problems below, use this sign where it is appropriate.

10. A stone block in the shape of a rectangular solid has a volume of $2\frac{1}{2}$ cubic yards. It weighs about 600 lbs. per cubic foot. What is its total weight?

11. An apartment house is built in the shape of a rectangular prism 210 ft. long, 30 ft. wide, and 30 ft. high. How many cubic feet of space is there in the building? (Ignore the thickness of the walls.) Express the volume also in cubic yards.

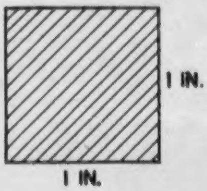
12. An electric fan is advertised as moving 3375 cubic feet of air per minute. How long will it take the fan to change the air in a room 27 ft. by 25 ft. by 10 ft.?

13. Imagine a building in the shape of a cube $\frac{1}{16}$ mile on a side. (How does $\frac{1}{16}$ mile compare with the length of a football field?) If people in the building insist that there shall be 10 cubic yards of space for each person, how many people will the building accommodate?

14. A large sandbox with a base 10 ft. long and 9 ft. wide is built in a park. A dump truck carrying five cubic yards of sand is emptied into the box. If the sand is leveled off, what is its depth? Give the answer both in feet and in inches.

15. A pirate's treasure chest was dug up and found to be filled with gold. The chest was a rectangular prism 2 feet 6 inches long, 18 inches wide, and 1 foot deep. Allowing for air spaces between the gold pieces, we can suppose a cubic foot of gold weighs 600 lbs. Could five men, each of whom can lift 400 lbs., lift the chest out of the hole?

16. An iron rod is to be made with a square cross section 1 inch on a side. That is, the end looks like this. If a cubic foot of iron were molded into this shape, how long would the rod be? Express the answer in inches, in feet, and in yards.



*17. If a building were a one-mile cube, that is a cube one mile on an edge, would there be space for the whole population of the United States (about 176,000,000)? For the population of China (about 400,000,000)? Could you accommodate the population of both countries together? (Allow 10 cubic yards per person.)

*18. The length, width, and height of a rectangular prism are measured as $10\frac{1}{2}$ inches, $5\frac{0}{2}$ inches, and $3\frac{1}{2}$ inches. Find the volume of its interior from these measurements.

*19. If the measurements in Problem 18 are understood to have a precision of $\frac{1}{2}$ inch (i.e., to the nearest half inch) then:

(a) The true length is between __?__ in. and __?__ in.

(b) The true width is between __?__ in and __?__ in.

(c) The true height is between __?__ in. and __?__ in.

*20. (a) The smallest rectangular prism which could be described by the measurements of Problem 18 would be __?__ inches by __?__ inches by __?__ inches.

(b) The largest rectangular prism which could be described by the measurements of Problem 18 would be __?__ inches by __?__ by __?__ inches.

(c) The volume of this smallest prism is __?__.

(d) The volume of the largest prism is __?__.

(e) The difference between the smallest possible volume and the answer of Problem 18 is __?__.

(f) The difference between the largest possible volume and the answer of Problem 18 is __?__.

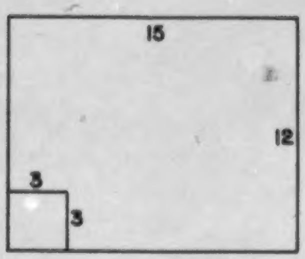
(g) The greatest possible error in the answer to Problem 18 is therefore __?__.

(h) Write the answer to Problem 18 in the following form:

The volume is $\underset{\text{answer to Problem 18}}{\underline{\qquad\qquad}} \pm \underset{\text{greatest possible error}}{\underline{\qquad\qquad\qquad}}$ cubic inches.

Problems 18, 19, 20 show, as in the case of area, that the possible error in a volume obtained after multiplying approximate measures is generally much greater than the form of the answer suggests. We must remember that in writing the answer to Problem 18 as $183\frac{3}{4}$ cubic inches, we do not mean that this is correct to the nearest quarter of a cubic inch. In fact, the error may be almost 28 cubic inches.

Probably the simplest possible figure not contained in a plane is the rectangular prism. Of course, as time goes on we shall learn to find volumes of other solids determined by such figures as other prisms, cones, cylinders, spheres, etc., but it is clearly a losing battle to try to consider individually all possible shapes. Often, of course, we can obtain the results by adding or subtracting volumes we already know about. For example, suppose that in a room 15 feet long, 12 feet wide, and 8 feet high, a man builds a closet in one corner which runs to the ceiling and has a base 3 feet on a side, so that the floor plan looks like the figure on the following page.

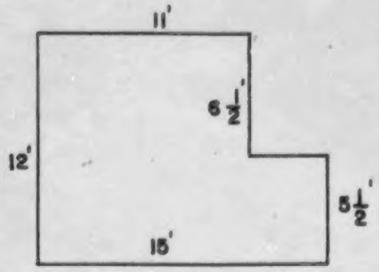


How do you find the volume of the remaining space in the room? What is it?

Suppose, however, that we are dealing with an awkward shaped solid which does not seem to be made up of rectangular prisms. To be definite, suppose you have picked up an odd shaped stone beside the road, and you want its volume. See if you can devise one or more methods of finding this volume.

Exercises 8-2d

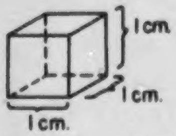
1. Measure the volume of a stone or other irregular object by a method you have devised.
2. The floor plan of a room is as shown:



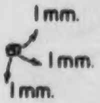
How many square feet of wall-to-wall carpeting would be necessary for the floor? What is the volume of the interior of the room if it is 9 feet high?

3. A pantry, the floor of which is 4 ft. by 5 ft., is 9 ft. high. It contains a deep freeze which is 2 ft. by 3 ft. by 7 ft. How many cubic feet of space are left in the room? Express the answer also in cubic yards.

In the preceding exercises you calculated surface areas in terms of square yards, square feet, and square inches. You calculated volumes in terms of cubic yards, cubic feet and cubic inches. Some metric units for measuring volumes are the cubic meter, cubic centimeter and cubic millimeter. The diagram at the left below is a picture of a cubic centimeter. The diagram at the right is a picture of a cubic millimeter.



1 cubic centimeter



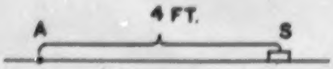
1 cubic millimeter

Exercises 8-2e

1. How many cubic millimeters in 1 cubic centimeter?
2. How many cubic centimeters in 1 cubic meter?
3. How many cubic millimeters in 1 cubic meter?
4. Draw a cube 3 cm. on each edge. Draw also a rectangular prism whose volume is 3 cubic centimeters. Which has the greater volume?
5. The length, width and height of a rectangular prism are 2 meters, 3 meters and 1 meter. Find its total surface area and its volume.
6. Suppose that in a room 5 meters long, 4 meters wide and 3 meters high, a man builds a closet in one corner. This closet runs to the ceiling and has a base 1 meter on a side. Find:
   (a) The volume of the room without the closet.
   (b) The volume of the interior of the closet.
   (c) The difference (a) - (b) above.
   (d) Is (c) the volume of the remaining space?

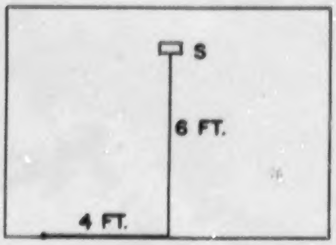
Dimension

Consider two flies sitting side by side at a point A by the baseboard of a room. One of them is trying to direct the other to a lump of sugar which is also by the baseboard. What directions does he need to give?



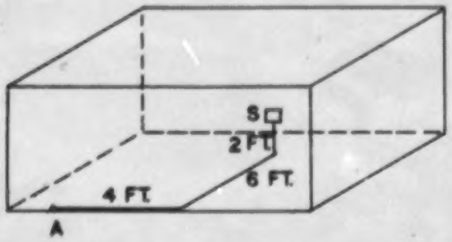
Presumably all he needs to say is "Follow the baseboard this way for four feet--you can't miss it!" The complete description of where the sugar is located by the baseboard can be given by one number and one direction. For this reason the edge of the room is called one-dimensional. Of course, the section of baseboard followed may be a single segment, or may turn one or more corners, so any segment or simple closed curve is one-dimensional.

If the lump of sugar S is somewhere out in the middle of the floor, this presents more of a problem to the directing fly. His friend cannot get there at all by following the baseboard. How can he give directions? One of the simpler ways would be as shown below.



"Follow the baseboard for four feet. Then turn to the left so you are headed perpendicular to the baseboard and crawl for six feet." In this case when the lump of sugar was interior to the rectangle it was found convenient to use two numbers and two directions parallel to an edge of the room to describe its location. For this reason the set interior to a rectangle (or any simple closed curve) is called two-dimensional.

If the lump of sugar is not on the floor at all but is somewhere else in the room, for example, suspended from the ceiling by a string, the problem of direction is harder still.



The directions then might go like this, "Crawl along the baseboard for four feet, then along the floor perpendicular to the baseboard for six feet. You will then be directly under the sugar. To get to the sugar fly directly up for two feet." This time we have used three numbers and three directions parallel to an edge of the room to describe how to get to the point S, so the interior of the room (that is, the interior of a rectangular solid) is called three-dimensional.

On the basis of the above discussion, what dimension would you want to give a point?

Roughly, the dimension of the set where the fly is shows how much freedom of motion he has. If he must stay in the one-dimensional set consisting of the floor's edge, he can only move one way or the other along this edge. If he may go anywhere in the two-dimensional set inside the rectangular edge he can crawl all over the floor. If he is merely confined to the three-dimensional set interior to the room, he can fly all over the room.

8-3. Other Measures

In our discussion of volume thus far we have used the units of volume related to linear measure, such as the cubic inch, cubic yard, or cubic mile. In practice we often use other units of volume. If you go to the grocery, you ask for milk, cream, or vinegar in quarts or pints rather than in cubic feet or cubic inches. Similarly you may ask for a bushel of peaches. Of course, there are definite relationships among these various measures and the cubic foot or cubic inch. You can find these relationships approximately. Take a quart jar and a rectangular container. Empty a certain number of quarts of water (or sand) into the container and find the volume in cubic inches by measuring the length, width, and the depth of water or sand. If possible, use a similar method to find the volume of a bushel basket. This time it will probably be better to take a container whose volume you already know in cubic inches or cubic feet and see how many of these will go into a bushel. Also, you will probably have to use something other than water as waterproof bushel baskets are scarce!

There would be an advantage in eliminating most of these unnecessary units of volume. (Indeed, this is one of the great advantages of the metric system about which you will hear a great deal as time goes on.) Since these units are in everyday use, we should know their relationships, or at least where to look them up. Unfortunately, in our English system we even use different units (sometimes with the same names) for measuring liquid and dry quantities. The quart measure whose volume you determined is actually the liquid quart, but there is a dry quart which is somewhat larger. For convenient reference, there is a section with information about the various units of length, area, volume, and weight and their relationships, which is placed at the end of this chapter.

Exercises 8-3a

1. Milk often comes in quart cartons that measure 7 in. by 3 in. by $2\frac{3}{4}$ in.
   (a) How many cubic inches is this?
   (b) Is this $\frac{1}{4}$ of a gallon?
   (c) What is sometimes done to this container to make it look larger?

2. A specific quart milk carton had these measures, in inches: $2\frac{3}{4}$ by $2\frac{3}{4}$ by $7\frac{1}{4}$. Would you have more or less than a quart in this carton? How much?
3. There is an old saying, "A pint's a pound, the world around." Give a reason why this is not necessarily true.
4. Berries are often sold in boxes that are labeled pints and quarts. A "quart" box measured $4\frac{3}{4}$ in. by $4\frac{3}{4}$ in. by $2\frac{7}{8}$ in.
   (a) How many cubic inches did it contain?
   (b) If a "dry" quart is $1\frac{1}{6}$ times the size of a liquid quart and a liquid quart contains $57\frac{3}{4}$ cu. in., how many cubic inches are there in a dry quart?
   (c) Does the measured "quart" box hold one dry quart?
5. A pint berry box measures $3\frac{3}{4}$ in. by $3\frac{3}{4}$ in. by $2\frac{1}{2}$ in.
   (a) How many cubic inches does this contain?
   (b) How many cubic inches should there be in a dry pint?
   (c) How much bigger or smaller is the box than it should be?
6. Is there any reason why a "dry" quart should be larger than a liquid quart?
7. A bushel of apples is priced at $3.25; apples can also be bought for 9¢ per pound. If a bushel holds 48 pounds of apples, how much do you save by buying a bushel?
8. A bushel of potatoes weighs 60 pounds. Which is cheaper, a bushel that costs $3.50 or 60 lbs. bought at 4 lb. for 25¢?
9. Half-gallon milk cartons have a base of $3\frac{1}{2}$ in. by $3\frac{1}{2}$ in. How tall should the carton be? If h stands for the number of inches in the height, write a number sentence for this problem.

*10. The number of inches in the edges of a certain rectangular prism are all whole numbers greater than 1. If the rectangular solid has a volume of one gallon, what are the measurements of the prism?

The measures in this chapter have dealt with space only. Lines, surfaces, volumes and angles have been measured. There are many things that are measured that have no connection with space. Temperature is one example; time is another. Everyone is familiar with weight as a measured quantity. The water, gas, and electricity used in your homes are measured. Can you name other things that are measured?

It is interesting to note that many items are measured by a scale marked on a line. Do you see that when we read a thermometer, we are really reading a number line? In order to read such measurements, we read the measurement of a line segment. Two of the items above are really measured by volume. Which ones are measured by volume? Are any measures read on a circular scale? Measurement involves much more than finding the length of line segments. Two of these measurements will be considered briefly, time and weight. As with all other kinds of measurement, the units used to measure these things must have the same nature as the thing measured. Time is measured by units of time, and weight is measured by units of weight. The scientific concepts of weight and time are complicated and will be left for later work in science. We will be concerned only with the use of some common units of weight and time.

Weight

Units of weight are used to describe how "heavy" a given volume of something is. It may seem funny to think of your body as a volume, but it does occupy some amount of space and it does have weight. A bushel of feathers does not have as much weight as a bushel of potatoes even though they occupy the same amount of space. The units of weight in the English system of measures are the ounce, the pound, and the ton. You should be familiar with the relations among these units.

16 oz. = 1 lb.
2000 lb. = 1 ton (t.)

The scientists who developed the metric system included units of weight and mass. Mass is, to a scientist, a measurement which is so much like that which we call weight that you may think of the two as being the same idea for the time being. You must be prepared to discard this idea some day, however, for the differences between the ideas of mass and weight are of great importance in the physical sciences. You will learn about these differences if you take a physics course in high school. Just to convince you that there is a difference we mention one of the more important of them. The weight of an object such as your body depends on its distance from the center of the earth. Your weight would be smaller on the top of Pike's Peak than it is in your home, and it would be very much smaller on a space ship as far from the earth as is, say, the moon. The mass of your body would be the same at any of these places, however. It does not depend on the place where it is measured.

It happens that the metric units for mass are more commonly used in describing an object than those for weight, and we will confine our discussion to them. Here the metric system has one big advantage over the English system because the units of mass are linked with volumes. The mass of one cubic centimeter of water at a specified temperature and pressure is the unit, one gram. This is especially convenient, for the unit of mass can be obtained at any time by anyone who can measure a given volume of water accurately.

An object whose mass is 1000 grams is said to have a mass of one kilogram. These are the only metric units of mass which will be used at this time.

1000 grams (gm.) = 1 kilogram (kg.)

Time

The basic unit of small divisions of time is the hour. Two smaller units are formed by dividing the hour into 60 equal parts, each one a minute, and 60^2 or 3,600 equal parts, each one a second. Longer periods of time are related to movements of the sun and moon. You should know the relations among the units: day, week, month, and year.

60 sec. = 1 min.
60 min. = 1 hr.
24 hr. = 1 day
7 da. = 1 week
$4\frac{1}{2}$ wk. $\approx$ 1 month
30 da. $\approx$ 1 month
52 wk. $\approx$ 1 yr.
365 da. $\approx$ 1 yr.
12 mo. = 1 yr.

It should be noted that the relations between day and month, week and month, week and year, and day and year are only approximate. This is in part due to the fact that the earth takes some extra hours and minutes in addition to 365 days in order to make a complete orbit around the sun.

Exercises 8-3b

1. Sea View School starts at 8:45 A.M. and closes at 3:30 P.M.
   (a) How many hours and minutes is this?
   (b) How many minutes are there in the school day?
   (c) If there are 8 equal class periods (including one for lunch) how many minutes in length is each class?
   (d) If there are only 7 equal class periods, how long is each period?
2. The Sea View School is open 188 days each year. How many hours is school in session each year? (See Problem 1.)
3. How many days are there between April 25 and May 6? Do not count either of these days.
4. How much time do you have each day for recreation if you sleep 9 hours, spend 6 hours and 45 minutes in school, study 2 hours, use 1 hour and 15 minutes for eating and help your mother 1 hour?

5. Water weighs approximately $62\frac{1}{2}$ lb. per cubic foot. What is the weight of the water in an aquarium that is 21 inches long, 18 inches wide and filled to a depth of 16 inches? (Hint: This is easier to work if all measurements are expressed in feet.)
6. Brand A of tomato juice has a label that says the weight is 1 lb. 12 oz., while brand B says the weight is 30 oz. Which can contains more tomato juice?
7. In a camp 70 people are fed at each meal.
    (a) If each person has a 6 oz. serving of tomato juice, how many ounces are needed?
    (b) How many pounds is this?
    (c) How many cans of Brand A juice would be needed? (See Problem 6.)
    (d) How many cans of Brand B?
    (e) If Brand A costs 42¢ a can and Brand B costs 44¢ a can, which would cost less to buy to serve one meal at the camp? How much less?
8. A ton of coal occupies approximately 35 cu. ft. of space.
    (a) How many tons will a coal bin hold that is 5 ft. by 8 ft. by 7 ft.?
    (b) How many pounds is this?
9. 1 cu. cm of water weighs 1 gram.
    (a) How many grams does one cubic meter of water weigh?
    (b) How many kilograms is this?
10. BRAINBUSTER: Which weighs more, a cubic foot of water or a cubic foot of ice? Why?

Computations with Measured Quantities.

Turn to page 268, 269 and re-read the paragraphs concerning the special meaning of "≐", "+", "-" when measurements are involved. Keep these ideas in mind when reading the text and working the problems which follow.

There are many times when it is necessary to add or subtract measures. Suppose John travels for 2 hours one day and 3 hours another day. We want to find John's total travel time. Another problem might be: There are 10 gallons of milk in a vat. If 3 gallons are removed, how much milk is left?

There are some situations which require further discussion. Suppose now that John's two travel times are stated as 1 hour the first day and 15 minutes the second day. Would his total travel time be (1 + 15) time units or 16 time units? Of course not! We can see readily that neither 16 hours nor 16 minutes would be his total travel time. The measurements must be expressed in the same units before an addition procedure is attempted. That is, we must express 1 hour as 60 min. or 15 min. as $\frac{1}{4}$ hr. before attempting to add.

It is silly to try to talk about an "addition" of 6 feet and 4 gallons! Think about why it is silly!

So far, when we have given different names to a measured quantity, we have expressed it wholly in terms of the same unit, such as a length which may be described by the name $3\frac{1}{2}$ ft. or by the name 42 in. This, of course, can always be done, but it is often more convenient not to do so. For example, we are more likely to speak of a length of 2 ft. 5 in. than $2\frac{5}{12}$ ft. or 29 in. though both of the latter are correct names for the length. Similarly, the duration of a train trip would more often be described as 2 hrs. 17 min. rather than by the names 137 min. or $2\frac{17}{60}$ hrs. which are also correct. Thus we shall here consider such composite names, realizing that we can always avoid them if we wish.

Let us now suppose that John travels for 2 hr. 40 min. the first day and 1 hr. 30 min. the second day. This time we do not want to express either travel time entirely in terms of hours or entirely in terms of minutes. Hence we must handle the two units separately.

$$\begin{aligned}\text{total travel time} &= (2 + 1)\text{ hr.}\quad (40 + 30)\text{ min.}\\ &= 3\text{ hr.}\quad 70\text{ min.}\end{aligned}$$

Since 70 min. is more than one hour, we can and should express it as 1 hour 10 min., so that

$$\begin{aligned}\text{total travel time} &= (3 + 1)\text{ hr.}\quad 10\text{ min.}\\ &= 4\text{ hr.}\quad 10\text{ min.}\end{aligned}$$

Here there were two separate additions and then a change of units. To leave 70 min. in the answer would be like carrying a pocketful of pennies instead of changing some of them to larger units like dollar bills. Some people call a change such as the last one in the problem above "simplifying the answer," or "expressing the answer in simple form."

Exercises 8-3c

Add and express the result in a simple form:

1. 3 hr. 18 min.
   6 hr. 52 min.

2. 5 yd. 1 ft. 10 in.
   6 yd. 5 in.
   4 yd. 1 ft. 8 in.

3. 3 gal. 3 qt. 1 pt.
   5 gal. 2 qt.
   6 gal. 1 qt. 1 pt.

4. 15 hr. 10 min. 50 sec.
   12 hr. 55 min. 18 sec.

5. 3 sq. yd. 5 sq. ft. 100 sq. in.
   2 sq. yd. 8 sq. ft. 47 sq. in.
   6 sq. yd. 7 sq. ft. 116 sq. in.

6. 3 cu. yd. 25 cu. ft. 500 cu. in.
   4 cu. yd. 19 cu. ft. 767 cu. in.
   2 cu. yd. 12 cu. ft. 1268 cu. in.

Subtraction, like addition, can be performed only when the units are the same, or can be changed so that they are the same. However, we sometimes find another difficulty in subtraction. Try this:

Subtract: 6 yd. 1 ft. 4 in.
2 yd. 2 ft. 9 in.

How can you subtract 9 inches from 4 inches or 2 ft. from 1 foot? We now use another important principle that you have used before. It is the reverse of the last step in addition. We change the form of the minuend without changing its value before we subtract. The work is shown here.

6 yd. 1 ft. 4 in. = 5 yd. 3 ft. 16 in.
2 yd. 2 ft. 9 in.
3 yd. 1 ft. 7 in.

The change in form is usually done by crossing out the original number and placing the new one above it. Tell how each of the new numbers was obtained. When more of the smaller units are needed, convert one of the next larger unit into the smaller ones.

Subtract and express the answer in a simple form:

7. (a) 41   3 ft. 1 in.
   19   8 in.

   (b) In both cases did you have to exchange one larger unit for its equal in smaller units? Explain.

8. 5 yd. 2 ft. 10 in.
   3 yd. 6 in.

9. 6 hr. 10 min.
   4 hr. 35 min.

10. 6 gal. 1 qt.
    3 gal. 3 qt. 1 pt.

11. 6 yd. 2 ft. 3 in.
    3 yd. 2 ft. 8 in.

12. 4 sq. yd. 2 sq. ft. 40 sq. in.
    3 sq. yd. 8 sq. ft. 105 sq. in.

13. 8 cu. yd. 6 cu. ft. 88 cu. in.
    2 cu. yd. 5 cu. ft. 99 cu. in.

Sometimes it is necessary to multiply measured quantities by number. Here is an example of a problem in which it is necessary. Webbing, 2 feet 8 inches long, is needed to make one belt for students on the safety patrol. If Lincoln school has 15 members on the safety patrol, how many yards of webbing are needed? We could put 2 ft. 8 in. down 15 times and add but it is much quicker to use the short-cut, multiplication. To multiply, multiply each unit separately and change the answer to a simple form, or the form that the problem asks for.

2 ft. 8 in.
15

30 ft. 120 in. = 40 ft. = $13\frac{1}{3}$ yd. 40 ft. is a simple form, but $13\frac{1}{3}$ yd. is the form that the problem asks for.

Multiply the measurement by the number indicated and express the answer in a simple form.

14. $\begin{array}{r} \text{6 hr. 18 min.} \\ \underline{5} \end{array}$

15. $\begin{array}{r} \text{5 yd. 2 ft. 27 in.} \\ \underline{19} \end{array}$

16. $\begin{array}{r} \text{2 gal. 3 qt.} \\ \underline{417} \end{array}$

17. $\begin{array}{r} \text{5 hr. 18 min. 35 sec.} \\ \underline{15} \end{array}$

18. $\begin{array}{r} \text{2 T 1689 lb.} \\ \underline{36} \end{array}$

19. $\begin{array}{r} \text{8 sq. ft. 127 sq. in.} \\ \underline{82} \end{array}$

When it is necessary to divide a measured quantity by a number it is usually easiest to change all measurements to the smallest sized unit, as was mentioned above before doing the division. After division simplify the answer.

Here is an example. Divide a line segment of length 3 yd. 2 ft. 8 in. into 8 equal parts.

$$\begin{array}{lr} \text{3 yd.} & = 108 \text{ in.} \\ \text{2 ft.} & = 24 \text{ in.} \\ \underline{\text{8 in.}} & \underline{= 8 \text{ in.}} \\ \text{3 yd. 2 ft. 8 in.} & = 140 \text{ in.} \end{array}$$

$$\begin{array}{r} 17\frac{1}{2} \\ 8\overline{)\,140} \end{array}$$

Therefore the length of each part = $17\frac{1}{2}$ in. = 1 ft. $5\frac{1}{2}$ in.

Divide each measurement by the number indicated and express the answer in a simple form.

20. $8\,\overline{)\ \text{6 hr. 16 min.}}$

21. $10\,\overline{)\ \text{23 yd. 1 ft.}}$

22. $32\,\overline{)\ \text{3 T. 496 lb.}}$

23. $11\,\overline{)\ \text{6 gal. 3 qt. 1 pt.}}$

24. $17\,\overline{)\ \text{4 sq. ft. 87 sq. in.}}$

25. $76\,\overline{)\ \text{2 cu. ft. 952 cu. in.}}$

Changing all measurements into terms of the smallest unit is not absolutely essential. You may find it interesting to try Problem 21 above without it.

There is a very interesting relationship between the work in this section and the discussion of different number bases. For example, suppose we chose to use the number base twelve. Then the expression 2 ft. 7 in. could be written at once as 27_{twelve} in. or $\frac{27}{10}_{\text{twelve}}$ ft. Note that in the base twelve, the number of square inches in a square foot would be 100_{twelve}. The number of cubic inches in a cubic foot would be 1000_{twelve}.

26. Give at least one example of familiar units in measured quantities which would be nicely expressed in the following number bases:
    (a) base two
    (b) base three
    (c) base sixteen
    (d) base sixty

27. BRAINBUSTER. An apple weighs $\frac{3}{4}$ oz. plus $\frac{3}{4}$ of its weight. How much does it weigh?

8-4. Summary

1. Units commonly used for measuring area and volume are related to the units for length.
2. If ℓ and w are the numbers of linear units in the length and width of a rectangle, the number of linear units in its perimeter may be found from the number sentence: $p = 2(\ell + w)$.
3. The number of square units, A, of area in a rectangle is the product of the numbers, ℓ, w, of linear units in the length and the width. Written as a number sentence, this statement becomes:

$$A = \ell w$$

Recall that when we found the area of a rectangle whose sides are 2 in. and 3 in. we did not multiply 2 in. by 3 in. to get this area. We used a unit of area, 1 sq. in., and determined the number of these units needed to cover the rectangular closed region. This number is obtained by multiplying the measures, 2 and 3, of the lengths of the sides. The area is then (2 × 3) square inches. A similar situation holds for volume.

4. The number, V, of cubic units of volume of a rectangular solid is the product of the number, B, of square units in the area of the base and the number, h, of linear units in the height. Written as a number sentence this statement becomes:

$$V = Bh$$

Since the base is a rectangle, this number sentence may also be written:

$$V = \ell wh$$

5. The same measurement may have many different names (for example 108 in., 9 ft., 3 yd., $\frac{3}{1760}$ mi.); one may be changed for another by making use of the ratios between the units of measure.
6. One-dimensional: one number and one direction needed to state the position of an object.
   Two-dimensional: two numbers and two directions needed to stat the position of an object.
   Three-dimensional: three numbers and three directions needed to state the position of an object.
7. Many quantities other than geometric are measured. Frequently this is done by reducing measurement to reading linear or circular scales, such as the kinds of measurement discussed in this chapter.

TABLES FOR REFERENCE

English Units

Metric Units

Measurements of Length

12 in. = 1 ft.
3 ft. = 1 yd.
16½ ft. = 1 rd.
320 rd. = 1 mi.
5280 ft. = 1 mi.

10 millimeter (mm.) = 1 centimeter (cm.)
100 cm. = 1 meter (m.)
1,000 m. = 1 kilometer (km.)

Measurements of Surface

144 sq. in. = 1 sq. ft.
9 sq. ft. = 1 sq. yd.
160 sq. rd. = 1 acre
43,560 sq. ft. = 1 acre
640 acres = 1 sq. mi.

100 sq. mm. = 1 sq. cm.
10,000 sq. cm. = 1 sq. m.
1,000,000 sq. m. = 1 sq. km.

Measurements of Volume

1728 cu. in. = 1 cu. ft.
27 cu. ft. = 1 cu. yd.

1,000 cu. mm. = 1 cc.
1,000,000 cc. = 1 cu. m.

Measurements of Weight

16 oz. = 1 lb.
2000 lb. = 1 t.

1,000 gram (g.) = 1 kilogram (kg.)
1,000 kg. = 1 metric ton

Liquid Measurements

16 fl. oz. = 1 pt.
2 pt. = 1 qt.
4 qt. = 1 gal.

1,000 cc. = 1 liter (ℓ.)

Dry Measurements

2 pt. = 1 qt.
8 qt. = 1 peck (pk.)
4 pk. = 1 bu.

Volumes are used for this.

Miscellaneous Measurements

1 gal. = 231 cu. in.
1 cu. ft. ≈ 7½ gal.
1 bu. ≈ 2150 cu. in.
1 dry qt. ≈ 1⅛ liquid qt.

Metric and English Equivalents

1 in. = 2.54 cm.
1 yd. ≈ 0.9 m.
1 mi. ≈ 1.6 km.
1 lb. ≈ 0.45 kg.
1 qt. ≈ 0.95 ℓ.

1 cm. ≈ 0.4 in.
1 m. ≈ 1.1 yd.
1 km. ≈ 0.62 mi.
1 m. ≈ 39.37 in.
1 kg. ≈ 2.2 lb.
1 ℓ. ≈ 1.05 qt.

[sec. 8-4]

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